Gaming on Coincident Peak Shaving: Equilibrium and Strategic Behavior

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Abstract

Coincident peak demand charges are imposed by power system operators or electric utilities when the overall system demand, aggregated across multiple consumers, reaches its peak. These charges incentivize consumers to reduce their demand during peak periods, a practice known as coincident peak shaving. In this paper, we analyze the coincident peak shaving problem through the lens of game theory, developing a theoretical model to examine the impact of strategic consumer behavior on system efficiency. We demonstrate that the game structure exhibits varying characteristics—concave, quasiconcave/discontinuous, or non-concave/discontinuous—depending on the extent of consumers' demand-shifting capabilities. For a two-agent, two-period setting, we derive closed-form Nash equilibrium solutions under each condition and generalize our findings to cases with multiple agents. We prove the stability of the equilibrium points and present an algorithm for computing equilibrium outcomes across all game scenarios. We also show that the peak-shaving effectiveness of the game model matches that of the centralized peak-shaving model but with increased levels of anarchy. In the cases of quasi-concave and non-concave game conditions, we analytically demonstrate in the two-agent setting that anarchy increases with consumers' flexibility and inequity, as measured by their marginal shifting costs, and we also analyze the influence of the number of agents on anarchy. Finally, we provide numerical simulations to validate our theoretical results.

Index Terms

Coincident peak shaving, Betting game, Price of anarchy, Equilibrium stability, Demand response, Power system operation

I. Introduction

The operation and economics of power systems are highly influenced by peak demand due to the real-time supply and demand balance requirements. Generator investments must cover peak demand, and operations need to consider the ramp rates induced by peak demand [1]. A notable example is the duck curve observed in California, USA, where solar energy generation creates a net load (or the demand remaining after subtracting variable renewable generation) valley during the daytime. However, because peak demand occurs at night and solar generation drops sharply, the net load increases dramatically and quickly—nearly 13,000 megawatts in three hours [2]. This rapid ramp rate and the need for higher total generation investment undermines the economics and efficiency of the power system.

Utilities and load-serving entities have long been providing solutions for peak shaving. A traditional approach is direct load control, where utilities sign contracts with consumers to gain access to large appliances and cycle them during peak periods, offering financial incentives to participating consumers [3]. However, this centralized method is intrusive and raises privacy concerns, making it less suitable in an era of distributed energy resources (DERs), such as storage units, smart appliances, and electric vehicles. With DERs, consumers become active participants in operations, strategically adjusting their electricity usage to minimize costs [4].

In addition to financial incentives, utilities have employed centralized pricing mechanisms, such as time-of-use (ToU) tariffs [5] and real-time tariffs [6], where the price structure is designed to achieve the utility's objectives. Although effective in shifting demand patterns, these methods are less focused on peak shaving, and the utilities' revenue recovery is dependent on the consumers' responsiveness [7], [8]. Another approach involves applying peak demand charges during system peak hours to encourage demand reduction [9], [10]. However, allocating peak demand charges among consumers remains a challenge, as the system peak demand does not align directly with individual consumers' peak demands [11].

Coincident peak (CP) demand charges present a promising alternative, where individual consumers are billed based on their demand during the system's peak time. These charges are incorporated into future electricity bills; for example, the 4CP mechanism in Texas adds charges from four hours of peak demand in the current year to the following year's electricity bills, potentially comprising up to 30% of an organization's monthly bill [12]. This mechanism provides strong incentives for consumers to reduce their demand during peak times. However, a significant challenge lies in the fact that the peak time is realized posteriorly. While many academic and industrial efforts focus on accurately predicting the system's peak time [13]–[15], they often overlook the interactions between consumers, whose collective demands determine the system peak. This gap naturally motivates a game-based framework and raises a critical question:

Whether the game-based framework is workable for the CP shaving problem, and how does consumers' strategic behavior in the game environment cause anarchy compared to the centralized method?

In this work, we address this question by formulating a theoretical game framework for the CP shaving problem. The framework is sufficiently rich to capture the salient decision-making behavior of individual consumers within the gaming environment, yet simple enough to allow for analytical characterization of their strategic decisions. Specifically, our contributions are as follows:

- We propose a CP game model with a fixed CP charge price and an individual penalty term for demand shifting, where all agents compete to determine the CP time and corresponding demand strategy. We show that the CP game model can exhibit concave, quasi-concave/discontinuous, or non-concave/discontinuous structures, depending on the agents' demand-shifting capabilities. In contrast, the widely used centralized peak shaving model is a trivial convex optimization problem.
- We analytically derive the Nash equilibrium (NE) for a two-agent, two-period setting under all CP game structures. Our analysis reveals that the NE is determined by the system's unbalanced demand and the agents' maximum shifting capabilities. By treating the system as a switched dynamic system, we prove that the equilibrium of the game model is globally uniformly asymptotically stable, regardless of switching, provided all agents' demands are non-negative. Furthermore, we demonstrate that a gradient-based algorithm with an update rule serving as a finite difference approximation of the asymptotically stable process can compute the equilibrium point.
- We analyze the impact of gaming agents' strategic behaviors by examining the peak shaving effectiveness and the price of anarchy (PoA). Our findings analytically indicate that the peak shaving effectiveness of the game model matches that of the centralized model but comes at the cost of economic inefficiency (anarchy), except for the concave game, where the outcomes are fully equivalent. For quasiconcave and non-concave games, we prove that the PoA increases with the inequality among agents, as measured by their marginal shifting costs. Additionally, under identical system conditions, we show that greater agent flexibility exacerbates PoA by altering the game type.
- Extending our analysis to multi-agent settings, we prove the unique NE under concave and quasi-concave conditions. For non-concave CP games, we demonstrate that system demand can still be balanced over two periods, deriving the NE for a subset of agents. We also establish the stability of the equilibrium point and validate the effectiveness of the gradient-based algorithm. The NE reveals equivalent peak shaving effectiveness between the game and the centralized model while showing that changes in the number of agents influence the game type and, consequently, the PoA.

A. Related works

To precisely position our study within the gap in the existing literature, we elaborate on our work from three perspectives closely related to the peak shaving problem: (i) Centralized peak shaving, (ii) CP demand charge, and (iii) Betting game.

Centralized peak shaving. Many studies address the peak shaving problem from the utility's perspective, formulating it as a centralized optimization problem aimed at minimizing costs or maximizing profits. In such models, pricing mechanisms, including tariffs or peak demand charges, are treated as decision variables [16]. This centralized formulation is straightforward to solve, as it typically results in a convex problem when the cost function is convex and the revenue function is concave [17]. Centralized approaches often rely on accurately modeling consumers' price response behavior to determine the optimal pricing scheme. This leads to a two-layer framework: the upper layer determines the price, while the lower layer models demand based on consumers' price response behavior [18]. The lower layer may involve constructing utility functions to represent consumer preferences [19], [20] or using data-driven methods to learn behavior from historical price and consumption data. However, capturing consumer price response behavior accurately in centralized models is challenging because consumer preferences are highly complex [21] and are influenced by dynamic environmental factors [22]. These factors are difficult to represent with a single model or dataset. Moreover, the centralized approach primarily focuses on reducing system peak demand and recovering utility revenue, often overlooking the individual benefits for consumers. This limitation motivates the adoption of a distributed approach that emphasizes consumer participation in peak shaving.

CP demand charge. The CP demand charge, which considers peak demand charges from both consumer and system perspectives, is an efficient method for peak shaving. Academic and industrial solutions for CP shaving generally focus on prediction and decision-making. Prediction involves estimating CP time as a probabilistic distribution, often using machine learning tools that leverage input features such as historical demand and weather conditions [14], [23]. Following this, decisions are made based on the prediction results. Industrial solutions primarily emphasize short-term historical data prediction, employing auto-regressive methods to iteratively update models and issue warning signals to consumers [15]. Academic solutions typically frame the problem as a scheduling task, making deliberate decisions based on CP time distributions. Examples include stochastic sequential optimization [24] and optimization with neural networks trained as decision policies [25]. Recently, an approach combining prediction and decision-making, termed 'decision-focused learning' or contextual optimization [26], has gained traction. This method trains weighting parameters using decision losses rather than prediction losses, aligning with downstream optimization tasks to produce effective decisions [27]. However, it has yet to be applied to the CP shaving problem. Moreover, existing studies often assume a single agent or 'price-taker' condition, neglecting the impact of agents' decisions on CP time predictions. This motivates us to incorporate consumer interactions as a critical aspect of CP shaving design.

Betting game. Without considering specific constraints, the CP shaving problem can be abstracted as players betting on discrete events, with winners sharing benefits based on their bids. This aligns with the classic concept of Pari-mutuel betting, where odds—analogous to peak times—vary with all betting strategies [28], [29]. A common application of Pari-mutuel betting is in sports events, where optimal betting strategies (size and target) are determined by solving discrete decision problems to maximize expected returns through the best picks combination [30], [31]. When multiple games occur simultaneously, the problem becomes an asset allocation problem, requiring combinatorial apportionment of resources [32], [33]. If time factors are introduced and the game is played sequentially, the problem evolves into a prediction market, where players iteratively gather information to refine decision policies [34], [35]. However, this is beyond our study's scope. These works provide valuable game frameworks that help establish a basic understanding of gaming in CP shaving. However, the constrained nature of our problem sets it apart from classic betting problems. Specifically, shifting demand negatively impacts consumer comfort and incurs profit loss, the total demand required to remain unchanged, and the outcomes of discrete events only depend on the strategies of all players, without any exogenous uncertainty. We fill the gaps in the literature by embedding these constraints into our model and formulating the CP shaving game framework.

The remaining of the paper is organized as follows: Section II introduces the model formulation and preliminary definitions, Section III analyzes the game property determined by agents' parameters and shows the Nash equilibrium structure under the two-agent setting, Section IV analyzes the stability of the equilibrium point and proves a global convergence of a gradient-based algorithm. Section V studies the impact of gaming agents' strategic behavior on peak shaving effectiveness and PoA. Section VI extends the two-agent setting to a multi-agent setting. Section VII provides the case study, and Section VIII concludes the paper.

II. Model and Preliminaries

In this section, we formulate the CP shaving game model and introduce the definitions. We consider two agents (consumers) gaming to reduce CP demand while maximizing their payoff in a two-period system. The CP game model is defined as $G = (N, \mathcal{X}, U)$, where

- $N = \{i, -i\}$ is the two-agent player set, and i, $-i$ are interchangeable. This notation is only for two agents, while in the multi-agent setting, which we will introduce in Section VI, $-i$ denotes all players but i.
- $\mathcal{X} = \mathcal{X}_i \times \mathcal{X}_{-i}$ is the strategy set formed by the product topology of each agent's strategy set $\mathcal{X}_i, \mathcal{X}_{-i}$, where $\times$ is the topology product;
- $U = \{f_i, f_{-i}\}$ is the payoff function set, where $f_i, f_{-i} : \mathcal{X} \to \mathbb{R}$.

During the game, agent i chooses strategy $x_i \in \mathcal{X}_i$ to maximize its payoff $f_i(x_i, x_{-i})$ with $x_{-i} \in \mathcal{X}_{-i}$, and the payoff is to minimize its two periods' costs,

$$\max_{x_i} f_i(x_i, x_{-i}) = -\pi(X_{i,1} + x_i)I(S_1(x) - S_2(x)) - \pi(X_{i,2} - x_i)I(S_2(x) - S_1(x)) - \alpha_i x_i^2, \quad \text{(1a)}$$

$$I(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad \text{(1b)}$$

$$S_1(x) = X_{i,1} + X_{-i,1} + x_i + x_{-i} = S_{1,0} + x_i + x_{-i}, \quad \text{(1c)}$$

$$S_2(x) = X_{i,2} + X_{-i,2} - x_i - x_{-i} = S_{2,0} - x_i - x_{-i}, \quad \text{(1d)}$$

$$x_i \in \mathcal{X}_i = \mathbb{R}, x_{-i} \in \mathcal{X}_{-i} = \mathbb{R} \quad \text{(1e)}$$

where $X_{i,1}, X_{i,2} \in \mathbb{R}^+$ are the baseline demand for agents i at time 1 and 2; x_i are the strategy (shiftable demand) for agents i, and we use x *as a vector* (without any index) to denote two agents' counterparts; S_1, S_2 are the system demands at time 1 and 2, which is a function of x, and in most cases, we omit the x when there is no ambiguity; $S_{1,0}, S_{2,0}$ is the system baseline demands; $\pi \in \mathbb{R}^+$ is a fixed CP charge price; $I(x)$ is the step function (indicator function). $\alpha_i \in \mathbb{R}^+$ is the penalty parameter for agent i when shifting demand, which expresses the comfort loss or any perceptive cost by shifting demand. Note that we focus on demand shifting and assume a constant energy rate besides the peak demand charge so we don't consider the energy cost.

It is obvious that the game is compact, convex, and bounded, where compact means each $\mathcal{X}_i$ is compact, convex means each $\mathcal{X}_i$ is convex, and bounded means each f_i is bounded. Note that we use a quadratic penalty to express a soft constraint to make the strategy set compact, and we have the linear correlation of the number of variables and agents. We then introduce the definitions of equilibrium and continuity.

Definition 1. We define the following: (1) *Pure-strategy Nash equilibrium (Nash [36]).* $(x_i^*, x_{-i}^*) \in \mathcal{X}$ is a NE in pure strategies of the game G if and only if (iff) $f_i(x_i^*, x_{-i}^*) \geq f_i(x_i, x_{-i}^*)$ for every $x_i \in \mathcal{X}_i, x_{-i} \in \mathcal{X}_{-i}$.

(2) *Upper semi-continuity (u.s.c.) and lower semi-continuity (l.s.c.).* The function $f_i : \mathcal{X} \to \mathbb{R}$ is called upper semi-continuous (u.s.c.) or lower semi-continuous (l.s.c.) if for every x_0 such that

$$\limsup_{x \to x_0} f_i(x) \leq f_i(x_0), \text{ or } \liminf_{x \to x_0} f_i(x) \geq f_i(x_0), \quad \text{(2)}$$

for all x in some neighborhood of x_0, respectively.

The step function makes the game analysis non-trivial. We thus separate the overall system into two subsystems 1 and 2, corresponding to CP time in 1 and 2, i.e., $\mathcal{X}_1 = \{x | S_1(x) \geq S_2(x)\}$ and $\mathcal{X}_2 =$

$\{x|S_1(x) < S_2(x)\}$, with the payoff functions

$$f_{i,1}(x_i) = -\pi(X_{i,1} + x_i) - \alpha_i x_i^2, x \in \mathcal{X}_1,$$
$$f_{i,2}(x_i) = -\pi(X_{i,2} - x_i) - \alpha_i x_i^2, x \in \mathcal{X}_2, \tag{3a}$$

It is obvious that each payoff function is concave, and by applying the first-order optimality condition [37], we have

$$x_i' = \arg\max_{x_i} f_{i,1}(x_i) = -\frac{\pi}{2\alpha_i}, x \in \mathcal{X}_1,$$
$$x_i' = \arg\max_{x_i} f_{i,2}(x_i) = \frac{\pi}{2\alpha_i}, x \in \mathcal{X}_2, \tag{3b}$$

With these results, we define critical points, balance points, and system average demand in the game model.

Definition 2. *Critical points, balance points, and system average demand.* We define the following:

1) *Critical point* r_i: the critical point of the payoff function $f_{i,2}$ described in (3), and it is easy to see the critical point of the payoff function $f_{i,1}$ is $-r_i$,
$$r_i = \frac{\pi}{2\alpha_i}, \tag{4a}$$
2) *Agent balance point* b_i: the demand difference of agent i in two time periods
$$b_i = \frac{X_{i,2} - X_{i,1}}{2}, \tag{4b}$$
3) *System balance point* b: the demand difference of the overall system in two time periods
$$b = b_i + b_{-i} = \frac{S_{2,0} - S_{1,0}}{2}. \tag{4c}$$
4) *System average demand* S: the average baseline demand of the overall system in two time periods is
$$S = \frac{S_{2,0} + S_{1,0}}{2}. \tag{4d}$$

The critical point indicates the maximum demand each agent can shift to avoid the CP charge within one subsystem. The agent balance allows each agent to shave its demand flat in two periods, which is the economic maximum demand shifting in the entire system. The reason is that once the agent balance point is reached, the CP charge is constant (half of the demand) regardless of the opponent's strategy. Thus, $\min\{r_i, b_i\}$ or $\max\{-r_i, b_i\}$ define the maximum shifting capability. The system average demand is useful to show the CP charge when the system demand is balanced in the two time periods. We then define the capable and non-capable agents.

Definition 3. *Capable and non-capable agents.* According to Definition 2, given agent i with baseline demand $X_{i,1}, X_{i,2}$, CP charge π, and penalty parameter α_i, the agent i is *capable* if it satisfy

$$-r_i \leq b_i \leq r_i, \tag{5}$$

Define the agent that didn't satisfy (5) as *non-capable* agent, including *upper non-capable* agent with $b_i > r_i$ and *lower non-capable* agent with $b_i < -r_i$.

According to Definition 2, (5) means agent i is economically capable of balancing its demand in the two periods during the game G to reduce the CP charge. The reason is that the agent's balance point b_i is within their critical point $-r_i, r_i$ of both subsystems, indicating that they will reach the balance point before the critical point when they shift demand.

III. Equilibrium analysis of the coincident peak game

In this section, we first introduce the definition of capable and non-capable agents, then show the properties of the two-agent two-period CP game G with the specific agent type, and prove the pure-strategy NE of the CP game.

A. Game properties

We first show the property of CP game G. Determined by the parameters of all agents, the game performs differently.

Proposition 4. *Concave, quasiconcave/discontinuous, and non-concave/discontinuous CP game.* Given critical point r_i, r_{-i} and balance point b_i, b_{-i}, b as define in Definition 2, the two-agent two-period CP game G satisfies one of the following:

1) *Concave CP game.* G is concave that all agents' payoff function $f_i(x_i, x_{-i})$ is concave in $x_i \in \mathcal{X}_i$ for each $x_{-i} \in \mathcal{X}_{-i}$ under the conditions of

$$b < -r_i - r_{-i}, \text{if } S_{1,0} \geq S_{2,0}, \tag{6a}$$

$$b > r_i + r_{-i}, \text{if } S_{1,0} < S_{2,0}, \tag{6b}$$

2) *Quasiconcave CP game.* G is quasiconcave/discontinuous that all agents' payoff function $f_i(x_i, x_{-i})$ is quasiconcave in $x_i \in \mathcal{X}_i$ for each $x_{-i} \in \mathcal{X}_{-i}$ [38] under the conditions of

$$-r_i \leq b_i \leq r_i; \tag{6c}$$

3) *Non-concave CP game.* G is non-concave/discontinuous if it is not concave or quasiconcave. The condition is

$$\{-r_i - r_{-i} \le b \le r_i + r_{-i}\} \cap [\{b_i < -r_i\} \cup \{b_i > r_i\} \cup \{b_{-i} < -r_{-i}\} \cup \{b_{-i} > r_{-i}\}]. \quad \text{(6d)}$$

Sketch of the proof. The payoff function is quadratic if only one subsystem is active during the solution process, indicating a concave and continuous game. The indicator function discontinues the game when the subsystem changes in the gaming process. According to the definition of the quasiconcave function, we separate three cases: 1) Agent i can switch the CP time before reaching their critical points in both subsystems 1 and 2; 2) Agent i can reach the critical point in subsystem 2 before switching, and the function is l.s.c. as defined in Definition 1; 3) Agent i can reach the critical point in subsystem 1 before switching, and the function is u.s.c. as defined in Definition 1. We show only the first case is able for both agents' functions to be quasiconcave. The non-concave condition is derived from the complementary set of concave and quasiconcave conditions. The detailed proof is provided in the appendix. □

This Proposition shows the game's properties depend on the relationship between agent's critical point and balance point, affected by baseline demand $X_{i,1}, X_{i,2}$, CP charge parameters π, and shifting penalty parameters α_i. In the concave game conditions with these game properties, all agents can't change the CP time together, and only one subsystem is active, determined by $S_{1,0}$ and $S_{2,0}$. If all agents are capable according to Definition 3, the CP game is quasiconcave. Otherwise, when both agents can change the CP time together, but one is non-capable, the CP game is non-concave. With these game properties, we then analyze the NE of the CP game G.

B. Nash Equilibrium

In this section, we study the NE of the two-agent two-period CP games in all conditions as described in Proposition 4. The main theorem is as follows

Theorem 5. *NEs in two-agent two-period CP game.* Given critical points r_i, r_{-i} and balance points b_i, b_{-i}, b as defined in Definition 2, the unique pure-strategy NE (x_i^*, x_{-i}^*) as defined in Definition 1 of the two-agent two-period CP game G satisfy one of the following:

1) *Concave CP game.*

$$x_i^* = -r_i, x_{-i}^* = -r_{-i}, \qquad \text{if } b_i < -r_i, b_{-i} < -r_{-i} - r_i - b_i; \quad \text{(7a)}$$

$$x_i^* = r_i, x_{-i}^* = r_{-i}, \qquad \text{if } b_i > r_i, b_{-i} > r_{-i} + r_i - b_i; \quad \text{(7b)}$$

2) *Quasiconcave CP game.*

$$x_i^* = b_i, x_{-i}^* = b_{-i}, \qquad \text{if } -r_i \leq b_i \leq r_i, -r_{-i} \leq b_{-i} \leq r_{-i}; \tag{7c}$$

3) *Non-concave CP game.*

$$x_i^* = -r_i, x_{-i}^* = b + r_i, \qquad \text{if } \{-r_{-i} - r_i \leq b \leq r_{-i} - r_i\} \cap \{b_i < -r_i \cup b_{-i} > r_{-i}\} \tag{7d}$$

$$x_i^* = r_i, x_{-i}^* = b - r_i, \qquad \text{if } \{-r_{-i} + r_i \leq b \leq r_{-i} + r_i\} \cap \{b_i > r_i \cup b_{-i} < -r_{-i}\} \tag{7e}$$

Sketch of the proof. As discussed in Proposition 4, the CP game G is possible to be concave, quasi-concave, and non-concave, which makes it hard to do systematic analysis. By the virtue of two-agent two-period setting, we can comprehensively analyze the NE in all possible conditions. We separate two cases according to the system CP time determined by $S_{1,0}, S_{2,0}$, and further separate the cases according to agents' own baseline demand $X_{i,1}, X_{i,2}$, then study whether both agents are capable, one of them is capable, and none of them is capable. The reason is that agents' best response is to shift demand away from the CP time when their demand is higher in the CP time, to avoid CP charges, or shift demand toward the CP time when their demand is lower in the CP time, to maintain the CP time. However, the shifting capability is determined by their their capability, i.e., the relationship between the critical point r_i and the agent balance point b_i. We analyze each conditions and show corresponding NE solutions. The details are provided in the appendix. □

This theorem shows the NE under each concave, quasiconcave, and non-concave CP game type. Note that the condition described in this theorem aligns with the conditions in Proposition 4, and we write out individual agent's conditions along with the NE solutions. Basically, the concave CP game includes two cases (7a) and (7b), both are aligned with (6a) and (6b) in Proposition 4. The condition of quasiconcave CP game (7c) is the same to (6c) in Proposition 4. The non-concave CP game conditions (7d) and (7e) are also aligned with (6d) in Proposition 4.

The NE also describes the relationship of gaming agents as fully cooperative (7a), (7b), fully competitive (7c), and mixed competitive and cooperative (7d), (7e). Specifically, When they are in a fully cooperative relationship, they cooperate to shift the demand away from the CP time. Once they can change the CP time together, they enter a competitive relationship where they compete with each other to change the CP time, but one agent's decision is limited by their shifting penalty. When all agents are capable, they are in a fully competitive relationship that balances their demand regardless of opponents' strategy. Indeed, if both agents want to be in a mixed relationship, they must be 'asymmetrical,' where their shifting penalty parameters should be different so that they have different shifting limitations. Otherwise, they are either fully cooperative or fully competitive.

We also show under the quasiconcave and concave game conditions, agents share the market equally, i.e., the strategy only depends on their own parameters, regardless of the opponent's strategy. While in the non-concave game, the agents' strategy depends on the opponent's strategy, and the agent with $x_{-i} = b \mp r_i$ is more flexible and dominates the other inflexible agent with $x_i = \pm r_i$ and benefit more through the competition. We denote agents' flexibility as the maximum relative demand shifts over their baseline demand difference, i.e., x_i^*/b_i. Here, x_i^* depends on the game type and is influenced by the shifting penalty parameter α, while b_i, as mentioned before, represents the economic maximum shifting capacity. Thus, flexibility is determined by both α and the baseline demand conditions $X_{i,1}$ and $X_{i,2}$.

IV. Equilibrium stability and algorithm convergence

Due to the indicator function in the model (1), the CP game system is a switched dynamics system. In this section, we first analyze the system stability and then develop a solution algorithm to reach the stable (equilibrium) point.

A. Equilibrium stability

There are two subsystem dynamics corresponding to CP time 1 and 2, separated by the indicator function and a switching logic between these two subsystems. According to [39], we consider a reasonable dynamic model in each subsystem in which each player changes his strategy following the gradient direction with respect to his strategy of his payoff function, then each player's payoff will increase given all other players' strategies. Denote the dynamic time index with k, and the gradient as $F_1(x) = [\nabla_i f_i(x), \nabla_{-i} f_{-i}(x)]^T$ with total differential operator ∇, the dynamics of subsystem 1 ($x \in \mathcal{X}_1$) is

$$\frac{dx}{dk} = \dot{x} = F_1(x) = [-(\pi + 2\alpha_i x_i), -(\pi + 2\alpha_{-i} x_{-i})]^T. \tag{8a}$$

The subsystem 2, i.e., $x \in \mathcal{X}_2$, follow the same structure with different gradient $F_2(x)$:

$$\frac{dx}{dk} = \dot{x} = F_2(x) = [\pi - 2\alpha_i x_i, \pi - 2\alpha_{-i} x_{-i}]^T. \tag{9a}$$

We then denote the CP game system with the switching logic as

$$\dot{x} = \begin{cases} F_1(x) & x \in \mathcal{X}_1 \\ F_2(x) & x \in \mathcal{X}_2 \end{cases}. \tag{10}$$

Our goal is to prove the overall CP game system with the switching logic is asymptotically stable. According to the NE described in Theorem 5, we call x^* as the equilibrium point of the CP game system (10) if one of the following is satisfied,

$$f_{i,1}(x_i^*) + f_{-i,1}(x_{-i}^*) = f_{i,2}(x_i^*) + f_{-i,2}(x_{-i}^*), \tag{11a}$$

$$F_1(x^*) = 0, S_{1,0} \geq S_{2,0}, \tag{11b}$$

$$F_2(x^*) = 0, S_{1,0} < S_{2,0}. \tag{11c}$$

Among them, the first condition (11a) corresponding to non-concave CP game (7d) and (7e), and quasiconcave CP game (7c), where both subsystems have the same CP charges and shifting penalty because the system demand is balanced in the two periods at NE and the shifting penalty is symmetric in the two time periods. The second and third (11b) and (11c) conditions correspond to the concave CP game (7a) and (7b), where the NE is obtained when the subsystem gradient F_1 or F_2 reaches zero. Then, we introduce the main stability results.

Theorem 6. *Global stability of equilibrium point.* The CP game system (10) is global uniform asymptotically stable in $\mathcal{X}_{\mathrm{s}}$, where

$$\mathcal{X}_{\mathrm{s}} = \{x | \alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 + \pi(S_{1,0} + x_i + x_{-i}) > 0 \cap \alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 + \pi(S_{2,0} - x_i - x_{-i}) > 0\}, \tag{12}$$

i.e., for every starting point $x \in \mathcal{X}_{\mathrm{s}}$, the solution $x(k)$ to the CP game system (10) converges to an equilibrium point $x^* \in \mathcal{X}_{\mathrm{s}}$ as $k \to \infty$, where x^* is the equilibrium point satisfy (11).

Sketch of the proof. The switched system property makes the stability analysis non-trivial. We separate two steps by first show each subsystem (8) and (9) is asymptotically stable in $\mathcal{X}$. Then, we include the switching logic and derive multiple Lyapunov functions with continuous properties in the switching surface to show the CP game system (10) is global uniform asymptotically stable in $\mathcal{X}_{\mathrm{s}}$. After that, we show the equilibrium (stable) point is obtained with $F_1 = 0$ or $F_2 = 0$ under the condition of concave CP game, and with $f_{i,1} + f_{-i,1} = f_{i,2} + f_{-i,2}$ under the condition of quasiconcave or non-concave CP game. The detailed proof is provided in the appendix. □

This Theorem shows the global uniform asymptotically stability of our CP game system in the strategy set $\mathcal{X}_{\mathrm{s}}$, indicating the CP game G can converge to the equilibrium points for the solution trajectory that within $\mathcal{X}_{\mathrm{s}}$. It is important to know that $\mathcal{X}_{\mathrm{s}}$ covers many scenarios in real operations, which guarantees the system stability to some extent. For example, the range that guarantees all agents' demands is non-negative $x_i \in [-X_{i,1}, X_{i,2}], x_{-i} \in [-X_{-i,1}, X_{-i,2}]$ is always within $\mathcal{X}_{\mathrm{s}}$.

With the stability property, our next step is to develop an algorithm to compute the equilibrium point.

B. Algorithm to determine equilibrium point

Given the finite difference approximation to the CP game system dynamics (10) with learning rate vector τ_h and gradient F_j

$$x_{h+1} = x_h + \tau_h F_j(x_h), j = 1, 2, \tag{13}$$

where j is the switching signal taking the value 1 for $S_1(x) \geq S_2(x)$ and 2 for $S_1(x) < S_2(x)$. This then forms a gradient-based algorithm following the updating rule (13) to gradually reach the equilibrium point. Among them, the gradient is determined by the subsystem on which the current solution lies, and the learning rate for each agent i $\tau_{i,h}$ is determined by its payoff function and gradient. Choosing a suitable learning rate is key to showing convergence performance, we provide the following Theorem to determine the learning rate.

Theorem 7. *Determination of equilibrium point.* Given the finite difference approximation as described in (13), a finite learning rate vector τ_h can be selected such that when $X_{i,1} + x_i \geq X_{i,2} - x_i$

$$-f_{i,1}(x_{\overline{h}}) < -f_{i,1}(x_{\underline{h}}), -f_{i,2}(x_{\overline{h}}) > -f_{i,2}(x_{\underline{h}}), \tag{14a}$$

when $X_{i,1} + x_i < X_{i,2} - x_i$,

$$-f_{i,1}(x_{\overline{h}}) > -f_{i,1}(x_{\underline{h}}), -f_{i,2}(x_{\overline{h}}) < -f_{i,2}(x_{\underline{h}}), \tag{14b}$$

where $F_j(x_h) \neq 0, j = 1, 2$; $\underline{h}$ and $\overline{h}$ is a switched pair that satisfies $\underline{h} < h < \overline{h}$ and $\underline{h} = \overline{h} = j$, $h \neq j$. The same will also hold for agent $-i$.

Sketch of the proof. We use the backtracking line search method [37] to calculate the learning rate, which is affected by the subsystems of the current and future steps. For the concave game, only one subsystem will be active during the entire solution process, according to the backtracking line search, the gradient F_1 or F_2 will reduce gradually, and combined with Theorem 6, the gradient will reduce to zero and reach the equilibrium point satisfy (11b) or (11c).

When switching happens, agents' individual peak times are different, and we analyze the objective function $-f_{i,1}, -f_{i,2}$ change following the backtracking line search criteria by judging where the subsystems lie in the current and next steps. Given a trajectory starts from subsystem 1 at h, switches to subsystem 2 at $h+1$, and back to subsystem 1 at $h+2$, suppose agent i's individual peak time is 1, we then show agent i update its decision x_i when switching from subsystem 1 to 2, and agent $-i$ update its decision x_{-i} when switching from subsystem 2 to 1, i.e., the $\tau_{-i,h} = 0, \tau_{i,h+1} = 0$, correspondingly, agent i's objective $-f_{i,1}$ decrease and agent $-i$'s objective $-f_{-i,2}$ decrease. The reason for the change is that their individual peak time aligns with the system's CP time. Note that this learning rate is determined by their own payoff functions and gradients, allowing the gradient for agent i to increase its decision variable while the gradient for agent $-i$ keeps the decision variable. By the same analysis, we know agent i's objective $-f_{i,2}$ increase and agent $-i$'s objective $-f_{i,1}$ increase in the trajectory starting from subsystem 2, switched to subsystem 1, and back to subsystem 2. We then show these results still hold when the trajectory stays more steps in one system between two transitions. This proves the reduction of the distance between objective functions in the two time periods. Combined with Theorem 6, the entire system finally will satisfy (11a) and reach the equilibrium point. The details are provided in the appendix. □

This Theorem shows that the finite learning rate can be chosen in each step using the backtracking line search. Following this updating rule (13), combined with Theorem 6, a gradient-based algorithm can reach the equilibrium point. Specifically, the difference in each agent's payoff function in the two subsystems reduces gradually, and depending on whether the game is concave or not, the difference can be reduced to zero or until one of the subsystems' gradients reaches zero.

As Theorem 6 shows, the system is asymptotically stable in $\mathcal{X}_\mathrm{s}$, combined with Theorem 7, we know this algorithm computes the NE as described in Theorem 5. Because the switched system is not globally uniform asymptotically stable in $\mathcal{X}$, so we need to set the initial point appropriately. Note that we don't use a higher-order gradient descent method, such as Newton's method, because the fast updating with higher-order gradient information lets the solution in each subsystem converge to each subsystem's stable point too fast to realize the converge on the overall switched system.

V. Impact of Customers Strategic Behavior

In this section, we analyze gaming agents' strategic behavior in the two-agent two-period setting. First, we state a benchmark centralized peak shaving model for comparison, then analyze agents' strategic behavior from both an economic perspective with the price of anarchy (PoA), and a technical perspective with the peak shaving effectiveness.

A. Centralized peak shaving

We first state the centralized CP shaving model, which assumes a central operator has direct control over both agents in the formulated two-period condition to reduce the total cost of both agents, including the peak demand charge and the shifting cost. This centralized model thus maximizes the total objective function of both agents and is denoted as

$$x^* \in \arg\max_{x_i, x_{-i}} -f_i(x_i, x_{-i}) - f_{-i}(x_i, x_{-i}), \tag{15}$$

We now show in the following proposition that the centralized CP shaving model is equivalent to the centralized peak shaving model which minimizes the total peak demand of both agents.

Proposition 8. *Centralized peak shaving model*. The centralized CP shaving model (15) is equivalent to the convex peak shaving model as follows:

$$x^* \in \arg\min_{x_i, x_{-i}} \ \pi \max\{S_1(x), S_2(x)\} + \alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 \tag{16}$$

Proof. Take f_i, f_{-i} as defined in (1) in to (15), we have

$$\begin{aligned} f_i(x_i, x_{-i}) + f_{-i}(x_i, x_{-i}) &= \pi(X_{i,1} + x_i) I(S_1(x) - S_2(x)) + \pi(X_{i,2} - x_i) I(S_2(x) - S_1(x)) \\ &+ \pi(X_{-i,1} + x_{-i}) I(S_1(x) - S_2(x)) + \pi(X_{-i,2} - x_{-i}) I(S_2(x) - S_1(x)) + \alpha_i x_{-i}^2 + \alpha_i x_i^2 \\ &= \pi S_1(x) I(S_1(x) - S_2(x)) + \pi S_2(x) I(S_2(x) - S_1(x)) + \alpha_i x_{-i}^2 + \alpha_i x_i^2 \\ &= \pi \max\{S_1(x), S_2(x)\} + \alpha_i x_{-i}^2 + \alpha_i x_i^2. \end{aligned} \tag{17}$$

Note that the peak shaving model in (16) is convex because maximize two convex (linear) function $S_1(x), S_2(x)$ is convex; thus, we proves the Proposition. □

This model assumes direct control of each agent's demand and achieves peak shaving with minimal cost. Thus, we consider (16) as a benchmark for comparison of our CP game model and analyze the peak shaving effectiveness and PoA in the following sections.

B. Peak shaving ratio analysis

We show in the following theorem that the CP game can achieve the same effectiveness in reducing the system peak demand at its equilibrium.

Theorem 9. *Peak shaving performance of CP game*. The peak shaving effectiveness of the game model (1) at equilibrium is always 1, i.e.,

$$\frac{\max\{S_1(x^*), S_2(x^*)\}}{\max\{S_1(x^*_{\text{cen}}), S_2(x^*_{\text{cen}})\}} = 1, \tag{18}$$

for all $\pi, \alpha, X > 0$; where x^* is the game equilibrium results and x^*_{cen} is the centralized peak shaving results.

Sketch of the proof. We prove this Theorem by first analytically writing the solution for the centralized model shown in Proposition 8, and from Theorem 5, we have the game model solution. Then take the solutions of the centralized model and the game model under the same condition into the peak shaving effectiveness definition in this Theorem. The detailed proof is provided in the appendix. □

This Theorem shows the game model always reaches the same peak shaving performance when compared with the centralized model. The reason is that agents in both the game model and centralized model shift demand as much as possible to avoid CP charge without burdening more by their shifting penalty. Specifically, under the concave game conditions, both agents shifting capability is limited by their critical point, i.e., can't balance their demand, which is also true in the centralized model. Under quasiconcave and non-concave game conditions, the system demand in two time periods is balanced, same to the centralized model. However, although the overall peak shaving performance is the same, their individual demand shifting is different due to the information barrier, which reflects as cost of reaching the peak shaving performance. We then analyze the cost by showing the PoA in the following section.

C. *PoA analysis*

In this section, we study the PoA affected by gaming agents' strategic behavior in all three game structures, and we provide the following main results.

Theorem 10. *PoA with agent equity.* Given the PoA define as

$$P = \frac{f_i(x^*) + f_{-i}(x^*)}{f_i(x^*_{\text{cen}}) + f_{-i}(x^*_{\text{cen}})}, \tag{19a}$$

under the quasiconcave and non-concave game condition as described in Proposition 4, the PoA increases with the inequity among agents, as measured by the marginal shifting cost $\alpha_i x_i^*$, i.e.,

$$\frac{\partial P}{\partial[(\alpha_i x_i^* - \alpha_{-i} x_{-i}^*)^2]} > 0 \tag{19b}$$

where x^* is the game equilibrium result and x^*_{cen} is the centralized peak shaving results.

Sketch of the proof. From Theorem 5, we know the NE of the game model under quasiconcave and non-concave game conditions. Combined with the centralized model solution obtained from Theorem 9, we show the difference of nominator and denominator of the PoA as defined in (19a) can be expressed as a Euclidean distance between agents' marginal shifting cost $\alpha_i x_i^*$, i.e., $(\alpha_i x_i^* - \alpha_{-i} x_{-i}^*)^2$. The detail is provided in the appendix. □

Under non-concave and quasiconcave game conditions, although the CP charge is always the same due to the same peak shaving effectiveness, the shifting cost $\alpha_i x_i^{*2}$ increases due to the information barrier. We show in this theorem that the PoA increases with agent inequity, quantified by their marginal shifting cost $\alpha_i x_i^* = \partial(\alpha_i x_i^2)/\partial x_i^*$. Enhancing equity by balancing agents' marginal shifting costs emerges as an effective strategy to reduce the PoA, offering a pathway to design mechanisms that ensure both system effectiveness and fairness. For example. a large shifting penalty parameter and a high shifting amount together indicate a higher marginal shifting cost: the penalty represents greater comfort loss when shifting demand, while the high shifting amount stems from large demand differences requiring adjustment due to CP charging. Regulating these agents to shift less or offering incentives to lower their penalty parameter can balance marginal shifting costs, improve system equity, and reduce the PoA.

Theorem 11. *PoA with CP game type.* For given $\pi, S, \alpha > 0$, the PoA defined as Theorem 10 is highest at equilibrium for quasiconcave games, followed by non-concave games, and lowest for concave games, where it is always equal to 1, i.e.,

$$P(\text{Quasiconcave game}) \geq P(\text{Non-concave game}) \geq P(\text{Concave game}) = 1.$$

Sketch of the proof. We first show $P = 1$ is always true for the concave game due to the same solution structure between the centralized model and concave game model based on the results from Theorems 5 and 9. Then from Theorem 10, we know the PoA expression as Euclidean distance between agents' marginal shifting cost under quasiconcave and non-concave game conditions, indicating $P \geq 1$. Thus, quasiconcave games and non-concave games always cause higher PoA than concave games. We then show by fixing $\pi, S, \alpha > 0$, the agent's baseline demand $X_{i,1}, X_{i,2}$ can vary to cause different game structures, so as to different solution structure x_i^*, x_{-i}^* from Theorems 5. Combined with the fact that quasiconcave and non-concave games always balance system demand, i.e., $x_i^* + x_{-i}^* = b$ from Theorem 5, we prove the quasiconcave game solution shows a higher difference between x_i^*, x_{-i}^*, indicating higher PoA than a non-concave game. The detailed proof is provided in the appendix. □

This theorem demonstrates the impact of the CP game type on the PoA, where the game type reflects the agents' flexibility. We first fix the system conditions π, S while allowing the agents' conditions to vary. From Theorem 5, we know that agents' flexibility is determined by their shifting penalty parameters α and baseline demand $X_{i,1}, X_{i,2}$. To control the influence of both parameters, we also fix α, making the PoA dependent only on x_i^*, which reflects the difference in the game structure. As the balance point (demand difference) $b_i = (X_{i,2} - X_{i,1})/2$ increases, the agent's flexibility decreases, and the game type transitions from a quasiconcave game to a non-concave game, and finally to a concave game. This follows from the conditions $-r_i \leq b_i \leq r_i$ for both $i, -i$ to $\{-r_i - r_{-i} \leq b \leq r_i + r_{-i}\} \cap \{b \geq r_i + r_{-i} \cup b \leq -r_i - r_{-i}\}$ to $\{r_i + r_{-i} < b\} \cup \{-r_i - r_{-i} > b\}$ as stated in Proposition 4. Thus, the PoA increases with agents' flexibility and is reflected in the change in game type. It is also evident that fixing $X_{i,1}, X_{i,2}$ while allowing α_i to vary yields the same results. Note that this theorem also shows that under the concave game condition, the system's performance is always equivalent to that of the centralized model, indicating no harm to utility companies and agents when applying the concave CP game.

VI. Extension to Multi-agent CP Games

In this section, we consider an extension of our model from the two-agent two-period CP game to the multi-agent two-period CP game, and we analyze the NE, stability/convergence, as well as the consumers' strategic behavior.

A. Multi-agent CP game

We extend the CP game G from two-agent to multi-agent with $G' = (N, \mathcal{X}, U)$, such that,

- $N = \{1, 2, ..., |N|\}$ with index i is the player set, where $|\cdot|$ for a set means the number of elements inside a set (otherwise, it is absolute value); specifically, we slightly abuse the notation of $-i$ to indicate all players but i.
- $\mathcal{X} = \times_{i \in N} \mathcal{X}_i$ is the strategy set formed by the product topology of each agent's individual strategy set $\mathcal{X}_i$.
- $U = \{f_i | i \in N\}$ is the payoff function set, where $f_i : \mathcal{X} \to \mathbb{R}$.

Then agent i's payoff function as described in (1) can be generalized as follows:

$$\max_{x_i} f_i(x_i, x_{-i}) = -\pi(X_{i,1} + x_i) I(S_1(x) - S_2(x))$$

$$-\pi(X_{i,2} - x_i) I(S_2(x) - S_1(x)) - \alpha_i x_i^2, \tag{20a}$$

$$S_1(x) = \sum_{i \in N} (X_{i,1} + x_i) = S_{1,0} + \sum_{i \in N} x_i, \tag{20b}$$

$$S_2(x) = \sum_{i \in N} (X_{i,2} - x_i) = S_{2,0} - \sum_{i \in N} x_i, \tag{20c}$$

$$x_i \in \mathcal{X}_i = \mathbb{R}, \forall i \in N. \tag{20d}$$

Note that here we also slightly abuse the system demand notation S_1, S_2 denote the sum of N agent's demand at time 1 and 2.

The multi-agent CP game G' still follow the property as described in Proposition 4, i.e., the CP game G' is

1) Concave if

$$b < -\sum_{i \in N} r_i, \text{if } S_{1,0} \geq S_{2,0}, b > \sum_{i \in N} r_i, \text{if } S_{1,0} < S_{2,0}, \tag{21a}$$

2) Quasiconcave if

$$-r_i \leq b_i \leq r_i, \forall i \in N; \tag{21b}$$

3) Non-concave if

$$\{-\sum_{i\in N} r_i \le b \le \sum_{i\in N} r_i\} \cap \{\exists i \in N, b_i < -r_i \cup b_i > r_i\}. \tag{21c}$$

The concave and quasiconcave conditions are intuitive and can be obtained following the same process of Proposition 4. Then the non-concave condition is determined by the complementary set of concave and quasiconcave conditions.

In terms of the NE solutions, we can't analyze each agent's capability case by case like the two-agent setting. Thus, we separately analyze NE in each game types. We first show concave game has a unique NE as defined in Definition 1 according to [39], which is obtained when both agents reach their critical point as defined in Definition 2 corresponding to the active subsystems, and we have the NE as follows,

$$x_i^* = -r_i, i \in N, \text{if } b < -\sum_{i\in N} r_i, \tag{22a}$$

$$x_i^* = r_i, i \in N, \text{if } b > \sum_{i\in N} r_i. \tag{22b}$$

Noted that when all x_i^* take $-r_i$, the system must lie in subsystem 1 where $S_{1,0} \ge S_{2,0}$, and agents shift demand from time 1 to time 2, until reaching their critical point.

NE of the quasiconcave game is not intuitive and we introduce the following Proposition to analyze it.

Proposition 12. *Existence and uniqueness of NE in multi-agent quasiconcave CP game.* The quasiconcave CP game G' as described in (20) and satisfy (21b) has a unique pure-strategy Nash equilibrium (x_i^*, x_{-i}^*) as defined in Definition 1, where

$$x_i^* = b_i, i \in N, S_1(x^*) = S_{1,0} + \sum_{i\in N} x_i^* = S_{2,0} - \sum_{i\in N} x_i^* = S_2(x^*). \tag{23}$$

Sketch of the proof. To prove this Proposition, we derive two Lemmas showing the existence (Lemma 17) and uniqueness (Lemma 18). The basic principle of existence Lemma is based on [38], i.e., the sum of the player's payoff functions is upper semi-continuous in $x \in \mathcal{X}$ and game G' is payoff security, where we define in the proof. The proof of uniqueness Lemma is based on [39]. We first prove the uniqueness of NE in each subsystem 1 and 2. Then, we start with a two-agent setting to analyze all agents' best responses and use min-max formulation to show the NE of each agent. After that we apply this two-agent results to multi-agent setting by sequentially partition agents into two groups. The detailed proof is provided in the appendix. □

This Proposition shows the quasiconcave CP game G' has a unique NE and always obtains at all agents' balance point b_i. This shows when capable agents are gaming on the system, they will always balance their demand and also the system demand, reaching the best peak-shaving performance.

In terms of non-concave game, obviously, the NE is not unique, but we can still derive the NE structure. Note that in both concave and quasiconcave conditions, the NE of multi-agent CP game G' is the same as that of two-agent CP game G by simply extending agent $i, -i$ to $i \in N$. Following this idea, we also analyze the non-concave CP game G' based on the two-agent CP game G. Basically, Theorem 5 conveys an important message that the system demand will always be balanced in the two periods, i.e., $x_i^* + x_{-i}^* = b$, and one agent (less flexible according to shifting penalty parameters and baseline demand) first reach the critical point while the other agent balance the system demand in the two time periods.

Here we denote the agents whose baseline peak demand is in the system baseline CP time as *CP-time agent*, and the other as *non-CP-time agent*. We also call the set that includes all CP-time agents as CP-time agent set and includes all non-CP-time agents as non-CP-time agent set, denote as $N_{\text{cp}}, N_{\text{ncp}}$, respectively, such that $N_{\text{cp}} \cup N_{\text{ncp}} = N, N_{\text{cp}} \cap N_{\text{ncp}} = \emptyset$. We then provide the following Proposition to analyze the NE of non-concave multi-agent CP game G'.

Proposition 13. *NE in non-concave multi-agent CP game.* Given non-concave multi-agent CP game G' and $S_{1,0} < S_{2,0}$, under the condition of

$$\sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\} < \sum_{i \in N_{\text{cp}}} b_i, -\sum_{i \in N} r_i \leq b \leq \sum_{i \in N} r_i; , \tag{24a}$$

the NE solution for each CP-time agent and for the whole non-CP-time agent set is given by

$$x_i^* = \min\{r_i, b_i\}, i \in N_{\text{cp}}, \sum_{i \in N_{\text{ncp}}} x_i^* = b - \sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\}; \tag{24b}$$

otherwise, the NE solution for each non-CP-time agent and the overall CP-time agent set is given by

$$x_i^* = \max\{-r_i, b_i\}, i \in N_{\text{ncp}}, \sum_{i \in N_{\text{cp}}} x_i^* = b - \sum_{i \in N_{\text{ncp}}} \max\{-r_i, b_i\}, \tag{24c}$$

corresponding to the condition

$$\sum_{i \in N_{\text{ncp}}} \max\{-r_i, b_i\} > \sum_{i \in N_{\text{cp}}} b_i, -\sum_{i \in N} r_i \leq b \leq \sum_{i \in N} r_i; \tag{24d}$$

Sketch of the proof. The basic idea is to assume two virtual agents that have the same baseline demand and shifting penalty conditions with the CP-time agent set and non-CP-time agent set. However, their strategy structures are different, where the agent set's strategy is determined by the aggregation of each individual agent's strategy. From Theorem 5, we derive the best strategies of two virtual agents, and from the best response analysis, we know the agent sets and virtual agents have the same response rationale due to the same baseline conditions. Basically, they will both shift demand away from CP time or to CP time to make a profit, and the profit positively relates to their shifting amount. We then link the two virtual agents' strategies to the agent sets' strategy and show the equilibrium point for the CP-time agent set and the non-CP-time agent set under different baseline conditions. The detailed proof is provided in the appendix. □

In the Proposition, we present the condition for $S_{1,0} < S_{2,0}$; when $S_{1,0} \geq S_{2,0}$, the only difference is to replace $\pm r_i$ with $\mp r_i$. This Proposition demonstrates that we can analytically determine the equilibrium solution for one set of agents — either the CP-time agent set N_{cp} or non-CP-time agent set N_{ncp}. For the remaining set, we can determine only the aggregated performance of the set. Note that we can't get the analytical NE solutions for each agent because agents in the remaining set can internally adjust their strategies while maintaining the same aggregated (set-level) performance. This internal adjustment does not affect the existence of the NE. Essentially, more flexible agents, those with lower shifting penalty parameters and higher baseline demand differences, will shift more to achieve the overall shifting performance of their set.

We then analyze the stability of the equilibrium point under multi-agent CP game G' based on the two-agent game's results as described in Theorem 6. Following the same proof structure, we can derive the asymptotically stable property also holds for multi-agent CP game G', and the strategy set $\mathcal{X}_{\text{s}}$ change to include all agents' decisions as follows

$$\mathcal{X}_{\text{s}} = \{x| \sum_{i\in N} \alpha_i x_i^2 + \pi(S_{1,0} + \sum_{i\in N} x_i) > 0 \cup \sum_{i\in N} \alpha_i x_i^2 + \pi(S_{2,0} - \sum_{i\in N} x_i) > 0\}. \tag{25}$$

With this stability property, we then show that a finite learning rate in (13) can still be chosen such that a gradient-based algorithm using (13) as updating rule finds the equilibrium point for the multi-agent CP game G'. The only difference in the multi-agent setting is that the criteria shift from the convergence of each agent's objective across two time periods to the convergence of the aggregated objectives of all CP-agents and non-CP-agents across the two periods.

Remark 14. *Determination of equilibrium point in multi-agent CP game.* A finite learning rate τ_h as described in (13) can be chosen such that when $S_{1,0} \geq S_{2,0}$,

$$\sum_{i\in N_{\text{cp}}} -f_{i,1}(x_{\overline{h}}) < \sum_{i\in N_{\text{cp}}} -f_{i,1}(x_{\underline{h}}), \sum_{i\in N_{\text{cp}}} -f_{i,2}(x_{\overline{h}}) > \sum_{i\in N_{\text{cp}}} -f_{i,2}(x_{\underline{h}}). \tag{26a}$$

$$\sum_{i\in N_{\text{ncp}}} -f_{i,2}(x_{\overline{h}}) < \sum_{i\in N_{\text{ncp}}} -f_{i,2}(x_{\underline{h}}), \sum_{i\in N_{\text{ncp}}} -f_{i,1}(x_{\overline{h}}) > \sum_{i\in N_{\text{ncp}}} -f_{i,1}(x_{\underline{h}}). \tag{26b}$$

When $S_{1,0} < S_{2,0}$, the greater and less sign reverse.

To conclude, the game framework is also workable for the multi-agent two-period setting as the NE still exists and is asymptotically stable within a strategy set as defined in (25). Also, a gradient-based algorithm with the updating rule as (13) can still compute the equilibrium point. We then analyze the impact of consumers' strategic behavior in the multi-agent setting.

B. Consumers strategic behavior in Multi-agent CP game

We show the peak shaving effectiveness and PoA under the multi-agent setting in this section, where the centralized model could be denoted as $x^*_{\text{cen}} \in \arg\max_{x_i} \sum_{i \in N} -f_i(x)$. Combined with Proposition 8, under the concave game condition, the centralized solution can be easily obtained by the first-order optimality conditions.

The peak shaving effectiveness, as defined in Theorem 9, is still equal to 1 at the equilibrium of the game model. The reason is that the system demand will be balanced under all game conditions, the same as the centralized model. The PoA in the multi-agent setting is $P_N = \sum_{i \in N} f_i(x^*)/\sum_{i \in N} f_i(x^*_{\text{cen}})$. As the centralized model is convex from Proposition 8, and $x^*_{\text{cen},i}$ is the unique minimizer for the problem, we always have $P \geq 1$. Specifically, the concave game model remains equivalent to the corresponding centralized model and always has $P = 1$.

With the agent number increase, the PoA under quasiconcave and non-concave games will be affected due to the game type changes, and we have the following remark.

Remark 15. *Game type with agent numbers.* As N increases, the game structure will more likely be a non-concave game.

It is intuitive that the non-concave game type is more likely to appear as the agent number increases because the probability that all agents are capable or non-capable reduces exponentially. As we analyzed at Theorem 11, the agent's flexibility changes game types and thus causes different PoA. This means the PoA of a small system is more sensitive to the agents' flexibility, and a large system can handle flexible agents as the game type could change from quasiconcave to non-concave due to adding inflexible agents.

VII. Numerical Example

In this section, we use numerical simulations to show the CP game solution, and we show the numerical test aligns with the theoretical analysis as Theorem 5 and Proposition 13 shows. We set the CP charge price to $\pi = 1$, and separate three two-agent two-period cases and one multi-agent two-period case with different baseline demand and shifting penalty parameters.

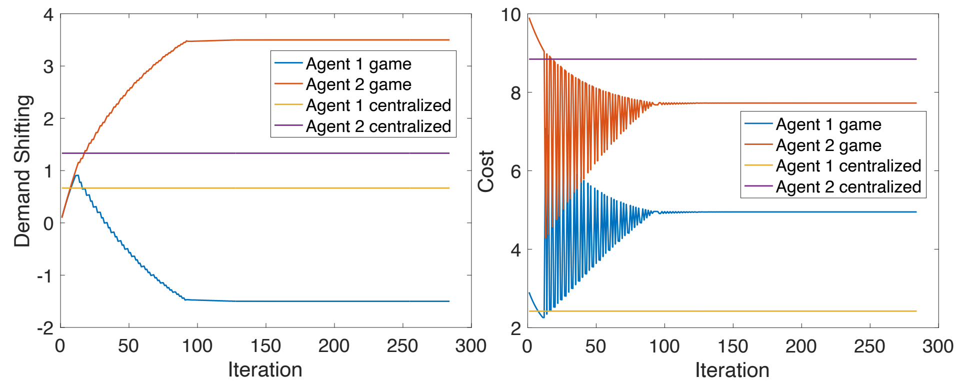

Fig. 1. Convergence under quasiconcave game condition.

A. Two-agent two-period CP game

We set the baseline demand as $X_{i,1} = 6, X_{i,2} = 3, X_{-i,1} = 3, X_{-i,2} = 10$, then change the shifting penalty parameters to change the agent capability.

(1) Set $\alpha_i = 0.2, \alpha_{-i} = 0.1$, then all agents' are capable according to Definition 3 and the behavior regime follow the quasiconcave game. The solution jumps between subsystems 1 and 2, so the cost function value jumps up and down, but the cost reduces or increases when jumping back to the last subsystem, i.e., the cost distance between these two subsystems reduces. Finally, the cost distance reduces to zero for each agent as they balance their demand in two time periods, and the system converges to the equilibrium point, corresponding to (7c). Compared with the solution from the centralized model, we observe that there are huge shifting changes and an increase in the system's overall cost due to anarchy, reflected as the PoA is 1.125, indicating anarchy increases the system cost by 12.5% compared to the centralized method. The peak shaving ratio is the same as they all balance system demand.

(2). Set $\alpha_i = 0.5, \alpha_{-i} = 0.1$, then agent i is non-capable, and agent $-i$ is capable according to Definition 3, and the behavior regime follow the non-concave game. The solution trajectory is similar to the quasiconcave condition, but the cost distance can't reach zero when convergence as both agents do not self-balance their demand in the two-time period, and the more flexible agent (agent $-i$) gets more benefits by lowering its demand in CP time. The results correspond to (7d). Compared with the quasiconcave game, we observe that the demand shifting from the game model is close to the centralized solution, and the system cost increase due to anarchy is also reduced with a PoA of 1.0941.

(3). Set $\alpha_i = 0.5, \alpha_{-i} = 0.6$, then both agents can't change the CP time together, and the behavior regime is concave game, where only subsystem 2 is active during the entire solution trajectory. Thus, the trajectory doesn't jump and gradually reduces to the critical point, which corresponds to (7b). Obviously,

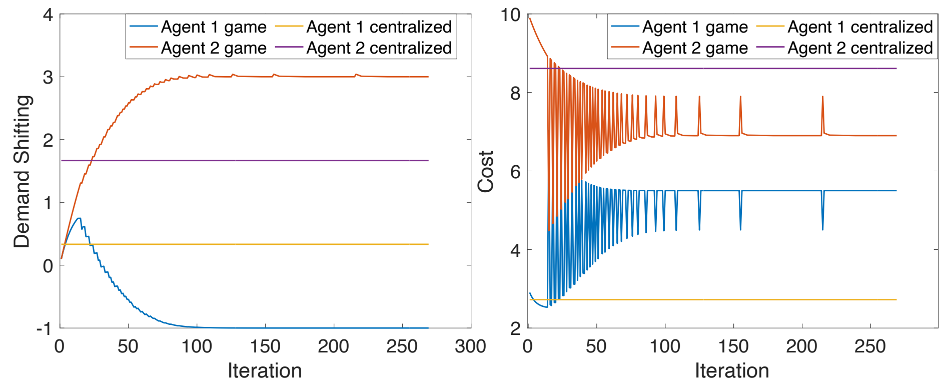

Fig. 2. Convergence under non-concave game condition.

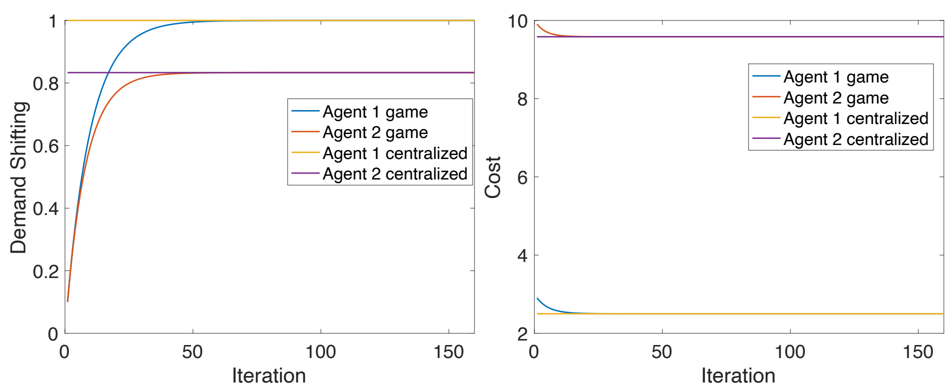

Fig. 3. Convergence under concave game condition.

the demand shifting converges to the same point with the centralized model, and the PoA is 1, indicating these two models are exactly equivalent.

B. Multiple agents and two-period CP game

We create a six-agent system with baseline demand and shifting penalty parameters, as Table I shows. We first notice the game is non-concave because all agents can change CP time together, but not all agents are capable, i.e., agents 3 and 4 are non-capable. Also, we know agents 1, 3, and 6 are non-CP-time agents, and agents 2, 4, and 5 are CP-time agents.

Figure 4 shows the convergence point and cost trajectory, and Figure 5 shows the cost trajectory from non-CP time agent and CP time agents' perspective. Noted that the results align with our analysis in Proposition 13 and Remark 14. Specifically, the non-CP time agents shift demand to the balance point or reach their critical point, i.e., agents 1 and 6 balance their demand in the two time periods, and

TABLE I

AGENTS' PARAMETERS AND SOLUTION OF GAME AND CENTRALIZED MODEL

Agent	1	2	3	4	5	6	Total
Baseline demand 1	7	3	10	1	2	5	28
Baseline demand 2	3	13	4	4	6	3	33
Penalty parameter	0.2	0.1	0.4	0.5	0.2	0.1	\
Centralized shifting	0.36	0.72	0.18	0.15	0.36	0.73	2.5
Game shifting	-2	3.85	-1.25	0.93	1.97	-1	2.5
PoA				1.1317			

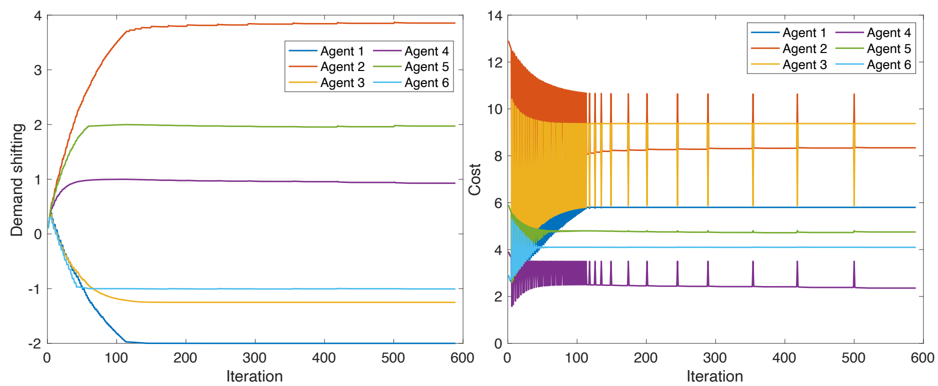

Fig. 4. Convergence of multiple agents concave CP game.

agents 3 reach the critical point limit by its shifting penalty. The CP time agents can internally shift their demand, which is observed after nearly 120 iterations, where CP time and non-CP time agents almost balance the system demand in the two time periods (Figure 5), agents 4 and 5, with the higher shifting penalty parameter than agent 2, gradually reduce its demand shifting, while agent 2's demand shifting increase. From the cost function trajectory (Figure 4), it is also clear that agent 4, with the highest shifting parameter, reduces more demand than agent 5.

Compared to the game model solution with the centralized model solution, obviously, the equilibrium solution deviates a lot, where all agents shift more demand due to the information barrier. This increases the cost and results in a PoA of 1.1317 while maintaining the same peak shaving performance.

C. Agent number impacts on PoA

In this section, we randomly generate agent samples for the systems with different agent numbers while satisfying the union of non-concave and quasiconcave game conditions as described in (21b) and (21c),

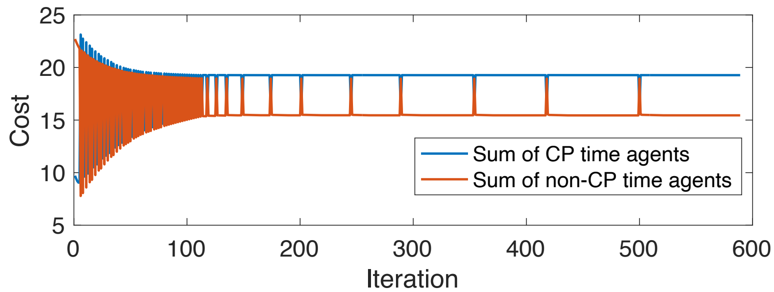

Fig. 5. Convergence of CP time agent and non-CP time agent.

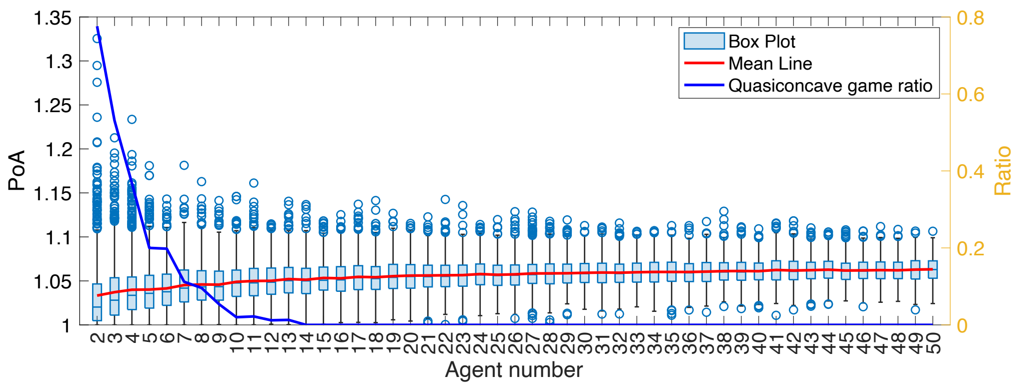

Fig. 6. Agent number impact on PoA. Each box contains 1000 samples, the circle denotes the outlier, the box upper line denotes the quartiles, and the short blue line inside the box is the median value.

i.e., $\sum_{i \in N} -r_i < b < \sum_{i \in N} r_i$. We set the agent's $i, i \in N$ baseline demand as $X_{i,1}, X_{i,2} \in (0, 15)$ and penalty parameters as $\alpha_i \in (0, 0.5)$. We loop the agent number from 2 to 50, and each agent number generates 1000 samples to calculate the PoA. We present the results in Fig. 6. As our theoretical analysis indicates, PoA is more fluctuates when agent number N is small; also, the game type is more varied with quasiconcave, concave, and non-concave, affected by agents' flexibility, indicating the small system is more sensitive to the agent's flexibility. When N is large, all game conditions become non-concave, and the mean PoA converges together, and the variance decreases. This means the large system is more stable and can eliminate the agent's flexibility influence on PoA.

VIII. Discussion and Conclusion

We propose a theoretical game framework to model the CP shaving problem. We show the game-based framework is workable for the CP shaving problem by analyzing the game structure and analytically deriving the NE for each game structure under the two-agent setting. We also prove the equilibrium points are global uniform asymptotically stable if all consumers' demand is non-negative and show that the gradient-based algorithm with an updating rule that serves as the finite difference approximation to the asymptotically stable process can compute the equilibrium points. We also extend our results to multi-agent settings. Using the equilibrium solution from two-agent settings, we analytically show that the gaming agents' strategic behaviors reach the same peak shaving effectiveness compared with an equivalent centralized peak shaving model but with a higher PoA under quasiconcave and non-concave conditions. We also show the same results still hold for the multi-agent setting. We conclude this is helpful for utility companies when applying the game model, especially for concave game conditions that are equivalent to centralized peak shaving.

By analyzing the quasiconcave and non-concave game PoA in the two-agent settings, we show PoA increases with the inequity level between agents measured by their marginal shifting cost. This implies an effective and equitable way to design CP shaving mechanisms is to balance their marginal shifting cost, i.e., offering some incentive to non-flexible agents with higher comfort loss when shifting demand or regulating them to shift less demand. We also show that the PoA increases as the game type transitions from a concave game to a non-concave game and then to a quasiconcave game, corresponding to an increase in agent flexibility. This suggests that greater agent flexibility amplifies system inefficiency. In the multi-agent setting, we show that the PoA is sensitive to agents' flexibility when the agent number is small as it changes the game type. With the agent number increase, the game is more likely to be a non-concave game, indicating the PoA of a large system is more likely to be stable. Combined with the numerical results, we show the marginal effect of adding additional agents diminishes as the agent number increases. Thus, forming a large system for flexible agents while a small system for inflexible agents is a good way to reduce system inefficiency.

Our work has a few limitations. First of all, we formulate the demand shifting penalty as a quadratic function. Future research is needed to examine more generic functions and even obtain empirical models from data-driven methods. Another limitation is that we only look at a scheduling problem with simultaneous decisions. Additional research is needed to study the CP game under a sequential context, where agents demand shifting decisions are made stage by stage given the non-anticipatory price and demand realization. This also motivates the last limitation, which is that we did not consider the incomplete information of the game model. In practice, agents' payoff structure should be private information that can't be observed by others, but agents may get others' previous decisions and system peak time to update their beliefs gradually. We would expect contextual optimization to be a promising way to infer the private payoff function from other observed features and embed it into the stochastic formulation.

APPENDIX A
PROOF OF PROPOSITION 4

Proof. We separate three conditions to analyze the continuity and concavity due to the indicator function.

(1) We first analyze the concave game conditions. To make the game concave, the CP time can't be changed, and there must be only one subsystem active in the entire solution process. In this case, the indicator function is eliminated, and the payoff function returns to a quadratic function as described in (3), which is concave and continuous. Because changing CP time requires the sum of all agents' demand shifting greater than the system balance point b, and the greatest demand shifting is their critical point. Thus, the conditions is $b < -r_i - r_{-i}$ when $S_{1,0} \geq S_{2,0}$ and $b > r_i + r_{-i}$ when $S_{1,0} < S_{2,0}$.

(2) We then prove the quasiconcave/discontinuous conditions. When the subsystem changes in the gaming process, it is obvious that the indicator function makes the game discontinuous. We then write the payoff function of agent i with the switching point $c_i = b - x_{-i}$

$$f_i(x_i, x_{-i}) = -(X_{i,1} + x_i)I(x_i - c_i) - (X_{i,2} - x_i)I(c_i - x_i) - \alpha_i x_i^2, \tag{27}$$

According to the definition of the quasiconcave function, for all $x_i', x_i'' \in \mathcal{X}_i$ and $\lambda \in [0, 1]$, agent i's payoff function $f_i(x_i, x_{-i})$ should satisfy the following for all $x_{-i} \in \mathcal{X}_{-i}$,

$$f_i(\lambda x_i' + (1-\lambda)x_i'', x_{-i}) \geq \min\{f_i(x_i', x_{-i}), f_i(x_i'', x_{-i})\}, \tag{28}$$

which is determined by the switching point c_i and the critical point r_i. There are three cases.

i) Agent i can switch CP time before reaching critical points in both subsystems 1 and 2, i.e., the switching point within the critical point $-r_i \leq c_i \leq r_i$, and we have

$$-r_i \leq b - x_{-i}, r_i \geq b - x_{-i}, \tag{29}$$

Graphically, the function is monotonically increasing before the switching point and decreasing after the switching point.

ii) Agent i can only switch CP time before reaching critical points in subsystem 1, i.e., the switching point is greater than the critical point of subsystem 2 $r_i < c_i$. Also, the function is l.s.c. as defined in Definition 1, i.e., $-(X_{i,2} - c_i) < -(X_{i,1} + c_i)$, and we have

$$r_i < b - x_{-i}, \frac{X_{i,2} - X_{i,1}}{2} = b_i < b - x_{-i}, \tag{30}$$

Graphically, the function is quadratic (concave) in the left part of the switching point, monotonically decreasing in the right part of the switching point, and the function is l.s.c. in the switching point.

iii) Agent i can only switch CP time before reaching critical points in subsystem 2, i.e., the switching point is less than the critical point of subsystem 1 $-r_i > c_i$. Also, the function is u.s.c. as defined in Definition 1, i.e., $-(X_{i,2} - c_i) < -(X_{i,1} + c_i)$, and we have

$$-r_i > b - x_{-i}, \ \frac{X_{i,2} - X_{i,1}}{2} = b_i > b - x_{-i}, \tag{31}$$

Graphically, the function increases monotonically in the left part of the switching point, is quadratic (concave) in the right part of the switching point, and is u.s.c. in the switching point.

Combining these three scenarios, we need both agents' payoff functions to satisfy one of them for all other agents' strategies to make the game quasiconcave. When agent i satisfy case i), according to the condition (29), agent $-i$'s strategy needs to satisfy $x_{-i} \in [b - r_i, b + r_i]$. From the graph description, we know agent $-i$'s strategy is $x_{-i} = c_{-i} = b - x_i$, $x_{-i} = -r_{-i}$, and $x_{-i} = r_{-i}$ under case i), ii), and iii), respectively. Obviously, r_{-i} and r_i belong to different agents and are independent, so agent $-i$ under case ii) and case iii) conditions can't guarantee the quasiconcave conditions. In case i), if $-r_i \leq b_i \leq r_i$, the agent can balance its demand in the two periods, thus the maximum $x_i = b_i$, if $b_i > r_i$ or $b_i < -r_i$, the best strategy is either r_i or $-r_i$. This means $x_i \in [-r_i, r_i]$, and thus $x_{-i} \in [b - r_i, b + r_i]$. Because both agents take the best strategy at the switching point, the only solution is $x_i = b_i, x_{-i} = b_{-i}$, and taking this into (29), we have the conditions

$$-r_i \leq b_i \leq r_i, \ -r_{-i} \leq b_{-i} \leq r_{-i}. \tag{32}$$

Otherwise, suppose agent i satisfies case ii), according to the condition (30), agent $-i$'s strategy needs to satisfy $x_{-i} < \min\{b - r_i, b - b_i\}$. We know agent $-i$'s strategy is $x_{-i} = -r_{-i}$ and $x_{-i} = r_{-i}$ under cases ii) and iii), and r_{-i} are independent to r_i and b_i. Thus, agent $-i$ under cases ii) and iii) can't guarantee the quasiconcave conditions. We then conclude that each player's payoff function is quasiconcave if (32) is true, which shows each agent is capable according to Definition 3.

(3) Other than the above two conditions, the game is non-concave and still discontinuous due to the indicator function. In this case, at least one agent is non-capable, and they can change the CP time together. We can directly write the conditions as the complementary set of concave and quasiconcave conditions with respect to $\mathbb{R}$, i.e., (6d). This finishes the proof of this proposition.

□

APPENDIX B
PROOF OF THEOREM 5

Proof. The basic idea of proving this Theorem is to analyze whether agents are capable, if they are not, whether they are upper non-capable or lower-non-capable, and different conditions corresponding to different NE. The rationale is that both agents' best response is to balance their demand in two periods, but this is limited by their baseline conditions, which is similar to analyzing the relationship between their critical point and balance point.

Note that the lower non-capable agent i satisfies $b_i < -r_i$, corresponding to scenario iii) as described in the proof of Proposition 4. Graphically, it shows the function increases monotonically in the left part of the switching point, and quadratic (concave) in the right part of the switching point, but not u.s.c. in the switching point. While the upper non-capable agent i is $b_i > r_i$, corresponding to scenario ii) as described in the proof of Proposition 4. Graphically, it shows the function is quadratic (concave) in the left part of the switching point, monotonically decreasing in the right part of the switching point, but not l.s.c. in the switching point.

We then separate many scenarios according to whether they are non-capable agents, upper non-capable, or lower non-capable, to analyze the NE and corresponding conditions.

(1) $S_{1,0} \geq S_{2,0}$, i.e., baseline CP time is 1.

(1a) $X_{i,1} \geq X_{i,2}, X_{-i,1} \geq X_{-i,2}$, i.e., both agents' individual demand is higher in the baseline CP time. We have four scenarios determined by whether they are capable or not.

- Both agents are capable agents, i.e., $-r_i \leq b_i \leq 0, -r_{-i} \leq b_{-i} \leq 0$, which means they can balance their demand, and the NE for all agent is their balance point,

$$x_i^* = b_i, x_{-i}^* = b_{-i} \tag{33a}$$

  and we have the conditions for each agent as

$$-r_i \leq b_i \leq 0, -r_{-i} \leq b_{-i} \leq 0. \tag{33b}$$

- Agent i is non-capable, while agent $-i$ is capable, i.e., $b_i < -r_i \leq 0, -r_{-i} \leq b_{-i} \leq 0$. Due to $X_{i,1} \geq X_{i,2}$, agent i must be lower non-capable, and the NE for agent i is the critical point, i.e., $x_i^* = -r_i$. Agent $-i$ can at least shift demand until the system demand balance in the two periods, and as agent i can't reach the balance point, agent $-i$ can save more by shifting more demand away from the CP time, but needs to compare with the penalty. Thus, the NE for agent $-i$ is

$$x_{-i}^* = \max\{-r_{-i}, y'_{-i}\}, y'_{-i} = b_{-i} - (x_i^* - b_i) = b - x_i^* \leq 0. \tag{34a}$$

  Then we compare the y'_{-i} with its critical point $-r_{-i}$. If $y'_{-i} < -r_{-i}$, i.e., $b - x_i^* = b + r_i < -r_{-i}$, which is equivalent to $-r_i - r_{-i} > b$, agent $-i$ can't change the CP time solely, then $x_{-i}^* = -r_{-i}$. The corresponding conditions for agent i is $-r_i > b_i$, and for agent $-i$ is

$$-r_{-i} \leq b_{-i} < -r_i - r_{-i} - b_i. \tag{34b}$$

where $-r_{-i} \leq b_{-i}$ is because agent $-i$ is capable. Otherwise, when $-r_i - r_{-i} \leq b$, the best strategy for agent $-i$ is y'_{-i}, and we have $x^*_{-i} = b + r_i$. The corresponding condition for agent i is still $-r_i > b_i$, for agent $-i$ is

$$b_{-i} \geq -r_i - r_{-i} - b_i. \tag{34c}$$

Note that this includes two scenarios when changing i to $-i$.

- Both agents are (lower) non-capable, i.e., $-r_i > b_i, -r_{-i} > b_{-i}$, and the NE is their critical point $x^*_i = -r_i, x^*_{-i} = -r_{-i}$.

(1b) $X_{i,1} \geq X_{i,2}, X_{-i,1} < X_{-i,2}$, i.e., agent i's demand is higher in the CP time, while agent $-i$'s demand is lower. Similar to (1a), when both agents are capable, the NE is their balance point. When they are not all capable agents, their optimal strategy is determined by whether they can change the CP time together and whether they can balance their own demand. Suppose agent i is non-capable, due to $X_{i,1} \geq X_{i,2}$, agent i must be lower non-capable, i.e., $b_i < -r_i$. Because CP time is 1 and agent i's demand is higher in time 1, although agent i is lower non-capable, it is possible to change the CP time solely without balancing its own demand. Thus, we separate two scenarios to analyze.

- Agent i can't change the CP time solely, i.e., $-r_i \geq b$. Agent i's best strategy is to shift demand away from the CP time, but can't reach the system balance point, so agent $-i$ will also shift some demand away from the CP time to reduce its cost but must compare with the shifting penalty. Thus, the NE is

$$x^*_i = -r_i, x^*_{-i} = \max\{-r_{-i}, b - x^*_i\}. \tag{35a}$$

When $-r_{-i} > b - x^*_i$, both agents can't change the CP time together, and the best strategy for agent $-i$ is $x^*_{-i} = -r_{-i}$. The condition for agent i is $b_i < -r_i$, and for agent $-i$ is $b_{-i} < -r_i - r_{-i} - b_i$. Otherwise, $-r_{-i} \leq b - x^*_i$, agent $-i$ can cooperate with agent i to change the CP time, and the best strategy for agent i is $x^*_{-i} = b - x^*_i = b + r_i$. The condition for agent i is the same as the lower non-capable condition $b_i < -r_i$, and for agent $-i$ is

$$-r_i \geq b, -r_i - r_{-i} \leq b, \tag{35b}$$

$$-r_i - r_{-i} - b_i \leq b_{-i} \leq -r_i - b_i. \tag{35c}$$

- Agent i can change the CP time solely, i.e., $-r_i < b$. Agent i shifts demand away from CP time and at least change the CP time, and agent $-i$ will move demand back to CP time to keep the original CP time. Thus, determined by which agent can shift more before reaching their critical point limit, the NE is

$$x^*_i = \max\{-r_i, b - x^*_{-i}\}, x^*_{-i} = \min\{r_{-i}, b - x^*_i\}. \tag{36a}$$

When $r_{-i} \leq b - x_i^*$, which means agent $-i$ first reach the critical point limit, then agent i must satisfy $b - x_{-i}^* > -r_i$. Thus, the best strategy is $x_i^* = b - r_{-i}, x_{-i}^* = r_{-i}$. The condition for agent i is the same as the lower non-capable condition $b_i < -r_i$, and for agent $-i$ is

$$b - r_{-i} > -r_i, b_{-i} > r_{-i} - r_i - b_i. \tag{36b}$$

Otherwise, we have the best strategy for both agents as $x_i^* = -r_i, x_{-i}^* = b + r_i$, and the corresponding condition for agent i is still $b_i < -r_i$, and for agent $-i$ is

$$-r_i - b_i < b_{-i} \leq r_{-i} - r_i - b_i. \tag{36c}$$

If agent $-i$ is non-capable, due to $X_{-i,1} < X_{-i,2}$, agent $-i$ must be upper non-capable, i.e., $b_{-i} > r_{-i}$ Here, we focus on the conditions that both agents can change the CP time together because we have shown the condition that they can't change the CP time above. Thus, the NE is determined by which agents can shift more before reaching their critical point limit, and we have the solution structure

$$x_{-i}^* = \min\{r_{-i}, b - x_i^*\}, x_i^* = \max\{-r_i, b - x_{-i}^*\}. \tag{37a}$$

Following the same process with (36), we know when $-r_i \geq b - x_{-i}^*$, agent $-i$ must satisfy $r_{-i} > b - x_i^*$, and the best strategy for both agents are $x_i^* = -r_i, x_{-i}^* = b + r_i$. The condition for agent $-i$ is the upper non-capable condition $b_{-i} > r_{-i}$, and for agent i is

$$r_{-i} \geq b + r_i, -r_i - r_{-i} \leq b, \tag{37b}$$

$$-r_i - r_{-i} - b_{-i} \leq b_i \leq -r_i + r_{-i} - b_{-i}, \tag{37c}$$

where the second condition is because both agents can change the CP time together. Otherwise, the best strategy for both agents are $x_i^* = b - r_{-i}, x_{-i}^* = r_{-i}$, and the condition for agent $-i$ is still the upper non-capable condition $b_{-i} > r_{-i}$, and for agent i is

$$-r_i \leq b - r_{-i}, r_i + r_{-i} \geq b, \tag{37d}$$

$$-r_i + r_{-i} - b_{-i} \leq b_i \leq r_i + r_{-i} - b_{-i}. \tag{37e}$$

(1c) $X_{i,1} < X_{i,2}, X_{-i,1} \geq X_{-i,2}$, i.e., agent $-i$'s demand is higher in the CP time, while agent i's demand is lower, which is the same to (1b) by changing the i to $-i$, and we omit the redundant math here.

(2) $S_{1,0} < S_{2,0}$, i.e., baseline CP time 2. Similar to the CP time 1 condition, we can still separate three cases with (2a) $X_{i,1} < X_{i,2}, X_{-i,1} < X_{-i,2}$; (2b) $X_{i,1} \geq X_{i,2}, X_{-i,1} < X_{-i,2}$; and (2c) $X_{i,1} < X_{i,2}, X_{-i,1} \geq X_{-i,2}$. The only difference in all cases is that when the best strategy is the critical point,

the critical point changes from $\pm r_i$ to $\mp r_i$. Other analysis is the same as (1), and we omit the redundant part.

To conclude, combining all the scenarios, we obtain the NE and corresponding conditions as (7) show and prove this Theorem. $\square$

APPENDIX C
PROOF OF THEOREM 6

Overview of the proof: The basic idea of proving this Theorem is to show that each subsystem is asymptotically stable in the strategy set $\mathcal{X}$, then, add the switching logic to show the system is global uniform asymptotically stable in $\mathcal{X}_\text{s}$ as described in Theorem 6, i.e., local uniform asymptotically stable in $\mathcal{X}$. We specify the proof process as follows:

- We first prove each subsystem (8) and (9) is asymptotically stable in the strategy set $\mathcal{X}$. (Lemma 16).
- Then we prove the overall system (10) with the switched logic is global uniform asymptotically stable in $\mathcal{X}_\text{s}$.

We first provide Lemma 16 to show the stability in the subsystem.

Lemma 16. *Subsystem stability at equilibrium point.* The system (8) is asymptotically stable in $\mathcal{X}$, i.e., for every starting point $x \in \mathcal{X}$, the solution $x(k)$ to the system (8) converges to an equilibrium point x^* as $k \to \infty$, where $F_1(x^*) = 0$.

Proof. The key is to show the rate of change of $\|F_1(x)\|^2$ is always negative for $F_1(x) \neq 0$ [39]. We have

$$\frac{dF_1}{dk} = G\frac{dx}{dk} = G\dot{x}, \tag{38}$$

where G is the Jacobian of $F_1(x)$, and $G = -2\text{diag}(\alpha_1, \alpha_2)$, where $\text{diag}(\cdot) : \mathbb{R}^2 \to \mathbb{R}^{2\cdot 2}$.

Now, according to (8a) and combining with the (38), we have

$$\frac{1}{2}\frac{d\|F_1^2\|}{dk} = \frac{1}{2}\frac{dF_1^T F_1}{dk} = F_1^T\frac{dF_1}{dk} = F_1^T G F_1 = \frac{1}{2}F_1^T(G + G^T)F_1. \tag{39a}$$

Because the $G + G^T$ is negative definite, we conclude that, for some $\epsilon > 0$, (39a) is equivalent to

$$\frac{1}{2}\frac{d\|F_1^2\|}{dk} \le -\epsilon\|F_1\|^2 \tag{39b}$$

Thus, $\lim_{k\to\infty}\|F_1\| = 0$, so that $x(k) \to x^*$, where x^* is the equilibrium point and $F_1(x^*) = 0$. Following the interior trajectory theorem from [39], we know $x^* \in \mathcal{X}$, which proves this Lemma by showing (8) is asymptotically stable. $\square$

From Lemma 16, the system (8) is asymptotically stable in $\mathcal{X}$, and this asymptotically stable result can be extrapolated to the system (9). We then add the switched logic to study global uniform asymptotically stability at the equilibrium point in $\mathcal{X}_{\rm s}$ and prove the Theorem.

Proof of Theorem 6. From Lemma 16, all individual subsystems are asymptotically stable in the strategy set $\mathcal{X}$. We then rewrite the gradient of two subsystems as follows,

$$F_1 = Ax + C_1, F_2 = Ax + C_2, \tag{40a}$$

$$A = [-2\alpha_i, 0; 0, -2\alpha_{-i}], C_1 = -[\pi, \pi]^T, C_2 = -C_1. \tag{40b}$$

We prove the Theorem based on the multiple Lyapunov function method [40]. Since our dynamic system is liner, the basic idea is to (i) find the functions $\mathcal{V}_j > 0$ in each subsystem $j = 1, 2$, for $\dot{\mathcal{V}}_j \neq 0$, the function $\mathcal{V}_j$ is always decreased along the solution of the jth subsystem in the region where this subsystem is active; (ii) on the switching surface $S_1(x) = S_2(x)$ the function $\mathcal{V}_j$'s value match.

We then choose $\mathcal{V}_j, j = 1, 2$ as follows:

$$\mathcal{V}_1 = -x^T \frac{A}{2} x - C_1^T x + d_1, \mathcal{V}_2 = -x^T \frac{A}{2} x - C_2^T x + d_2, \tag{41a}$$

$$d_1 = \pi S_{1,0}, d_2 = \pi S_{2,0}. \tag{41b}$$

Note that $-\partial \mathcal{V}_1 / \partial x = F_1(x)$ and $-\partial \mathcal{V}_2 / \partial x = F_2(x)$.

We first show the regions that guarantees function $\mathcal{V}_j > 0$

$$\alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 + \pi(x_i + x_{-i} + S_{1,0}) > 0, \tag{42a}$$

$$\alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 + \pi(-x_i - x_{-i} + S_{2,0}) > 0, \tag{42b}$$

which is the same as (12) shows in this Theorem. We then show the rate of change of $\dot{\mathcal{V}}_j$ is negative.

$$\dot{\mathcal{V}}_1 = \frac{\partial \mathcal{V}_1}{\partial x} F_1 = -((\frac{A}{2}^T + \frac{A}{2})x + C_1)(Ax + C_1) = x^T A' x + x^T B' - C_1^T C_1, \tag{43a}$$

$$A' = [-4\alpha_i^2, 0; 0, -4\alpha_{-i}^2], B' = -(\frac{A}{2}^T + \frac{A}{2})C_1 - A^T C_1 = [-4\pi\alpha_i, -4\pi\alpha_{-i}]^T. \tag{43b}$$

We have the critical point (equilibrium point) when $\partial(\dot{\mathcal{V}}_1)/\partial x = 0$

$$x' = -(A'^T + A')^{-1} B' = -[\frac{\pi}{2\alpha_i}, \frac{\pi}{2\alpha_{-i}}]. \tag{43c}$$

It is easy to see A' is negative definite. To show $\dot{\mathcal{V}}_1 < 0$ except the equilibrium point, we need to show $\dot{\mathcal{V}}_1(x') = 0$, so that other point must less than zero.

$$\begin{aligned}\dot{\mathcal{V}}_1(x') &= B'^T((A'^T + A')^{-1})^T A' (A'^T + A')^{-1} B' - B'^T((A'^T + A')^{-1})^T B' - C_1^T C_1 \\ &= -\frac{1}{4} B'^T (A'^{-1})^T B' - C_1^T C_1 = 0.\end{aligned} \tag{43d}$$

In terms of $\dot{\mathcal{V}}_2$, we have

$$\dot{\mathcal{V}}_2 = \frac{\partial \mathcal{V}_2}{\partial x} F_2 = -((\frac{A}{2}^T + \frac{A}{2})x + C_2)(Ax + C_2). \tag{44}$$

Due to $C_2 = -C_1$, the structure of $\dot{\mathcal{V}}_2$ is the same to $\dot{\mathcal{V}}_1$ and we can obtain $\dot{\mathcal{V}}_2 < 0$ except the equilibrium (critical) point $x' = [\pi/2\alpha_i, \pi/2\alpha_{-i}]$, where $\dot{\mathcal{V}}_2(x') = 0$.

Then, we show $\mathcal{V}_1 = \mathcal{V}_2$ on the switching surface $S_1(x) = S_2(x)$, where $x_i + x_{-i} = b$ and $f_{i,1} + f_{-i,1} = f_{i,2} + f_{-i,2}$, and the key is to show $-C_1^T x + d_1 = -C_2^T x + d_2$,

$$-C_1^T x + d_1 = \pi(x_i + x_{-i}) + \pi S_{1,0} = \pi b + \pi S_{1,0} = \frac{S_{2,0} + S_{1,0}}{2}, \tag{45a}$$

$$-C_2^T x + d_2 = -\pi(x_i + x_{-i}) + \pi S_{2,0} = -\pi b + \pi S_{2,0} = \frac{S_{2,0} + S_{1,0}}{2}. \tag{45b}$$

Then, we can conclude that $\mathcal{V}_j$ is always decrease except $\dot{\mathcal{V}}_j = 0$. Also, the rate of decrease of $\mathcal{V}_j$ along solutions is not affected by switching, and asymptotic stability is uniform with respect to j.

Now, let's analyze the convergent point. If the game is concave, only one subsystem j is active, i.e., subsystem 1 is active if $S_{1,0} \geq S_{2,0}$ and vice verse. Then the function value $\mathcal{V}_j$ decreases over time until $\dot{\mathcal{V}}_j = 0$, and the system reaches the stable point as (11b) and (11c) shows.

Otherwise, both subsystems will be active sequentially, and $\mathcal{V}_j$ will decrease over time until $\mathcal{V}_1 = \mathcal{V}_2$. The reason is that the dynamics from one subsystem always push the other subsystem active, e.g., F_1 in subsystem 1 always decreases x, which pushes the solution past the switching surface and actives subsystem 2. Thus, as $\mathcal{V}_1 = \mathcal{V}_2$ on the switching surface, both subsystems are finally stable on the switching surface. This means $x(k) \to x^*$ when $k \to \infty$, where x^* is the equilibrium point satisfy (11a). Thus, we prove the global uniform asymptotically stable of the overall system in (12).

□

APPENDIX D
PROOF OF THEOREM 7

Proof. The key to proving this Theorem is to select the learning rate based on the backtracking line search method. The learning rate depends on the subsystems on which the current and future steps lie, as well as the payoff functions and the gradients that each agent follows.

Suppose the current step is h, we expresses the backtracking line search condition for each agent i to choose the learning rate τ for the finite difference approximation (13) as follows,

$$-f_{i,j}(x_{i,h+1}) < -f_{i,j}(x_{i,h}) - \beta_1 \tau_{i,h} \|F_j(x_h)\|^2, \tag{46}$$

where β_1 is the parameter within $[0, 0.5]$; $f_{i,j}$ means the function can take either $f_{i,1}$ or $f_{i,2}$ determined by which subsystem the current and next step lies, and agent's decision is decoupled within each subsystem.

For example, if the current step $x_{i,h}$ lies in subsystem 1, j will take 1 for step h, and if the future step $x_{i,h+1}$ lies in subsystem 2, j will take 2 for step $h+1$. Note that $f_i(x)$ from (1) is formulated as a payoff (profit); here, we use $-f_i(x)$ to express the cost.

When selecting the learning rate, we gradually reduce $\tau_{i,h}$ by $\beta_2\tau_{i,h}, \beta_2 \in [0,1]$ until (46) satisfy. If the next step and current step lie on the same subsystem, this condition ensures the objective $-f_{i,j}(x_{i,h})$ reduces by at least $\beta_1\tau_{i,h}\|F_j(x_h)\|^2$. This proves the concave game convergence as all the steps lie in one subsystem, and the objective function is concave, thus, the objective function gradually reduces until $\|F_j(x_h)\| = 0$.

In terms of quasiconcave and non-concave games, there are switches during the algorithm iteration. It is easy to imagine that each agent shifts demand away from the baseline CP time at the beginning, monotonically reducing their costs, and the cost function is determined by the baseline CP time. Once they reach the balance point (switching surface), $x_{i,h} + x_{-i,h} = b$, the solution starts switching between two subsystems. Noted that when switching happens, both agents' individual peak time must be different, i.e., if $X_{i,1} + x_{i,h} > X_{i,2} - x_{i,h}$ for agent i, then $X_{-i,1} + x_{-i,h} < X_{-i,2} - x_{-i,h}$ must hold for agent $-i$; otherwise, there will be no switching;

We then write the difference between $-f_{i,1}(x_i)$ and $-f_{i,2}(x_i)$ as

$$-f_{i,1}(x_i) - (-f_{i,2}(x_i)) = \pi(X_{i,1} + x_i - (X_{i,2} - x_i)). \tag{47}$$

Suppose $X_{i,1} + x_{i,h} > X_{i,2} - x_{i,h}$ for agent i, we know

$$-f_{i,1}(x_{i,h}) > -f_{i,2}(x_{i,h}), -f_{-i,1}(x_{-i,h}) < -f_{-i,2}(x_{-i,h}). \tag{48}$$

Now, consider a trajectory starting from subsystem 1, switching to subsystem 2, and back to subsystem 1, i.e., $-f_{i,1}(x_h), -f_{i,2}(x_{h+1}), -f_{i,1}(x_{h+2})$, our goal is to show $-f_{i,1}(x_i)$ reduce while $-f_{i,2}(x_i)$ increase for agent i and $-f_{-i,2}(x_{-i})$ reduce while $-f_{-i,1}(x_{-i})$ increase for agent $-i$ through the trajectory. Due to the switching from subsystem 1 to 2, the gradient in subsystem 1 $F_1(x_h)$ must be negative to reduce the x_h so that the CP time changes. We choose the learning rate $\tau_{i,h}, \tau_{-i,h}$ such that

$$-f_{i,2}(x_{i,h+1}) < -f_{i,1}(x_{i,h}) - \beta_1\tau_{i,h}\|F_1(x_h)\|^2, \tag{49a}$$

$$-f_{-i,2}(x_{-i,h+1}) < -f_{-i,1}(x_{-i,h}) - \beta_1\tau_{-i,h}\|F_1(x_h)\|^2, \tag{49b}$$

As $X_{i,1}+x_{i,h} > X_{i,2}-x_{i,h}$ and from (48), we know (49a) is easy to be true and we use the corresponding $\tau_{i,h}$ to update $x_{i,h}$ following (46); while (49b) can't be true, so $x_{-i,h}$ will not be updated. Thus, the switching is caused by the update of agent i's decision, and we know $x_{i,h+1} < x_{i,h}$. Because the gradient is in subsystem 1, $F_1 < 0$, we know the $x_{i,h+1} > -r_i$ and suppose to be reduced to reach the critical point in subsystem 1 until converge, which indicates the right part of the critical point in the objective function $-f_{i,1}$, where its gradient $-f'_{i,1} > 0$. Thus, we have $-f_{i,1}(x_{i,h+1}) < -f_{i,1}(x_{i,h})$.

In subsystem 2, according to the trajectory, the gradient will push the solution back to subsystem 1, which requires x increase, and thus, we know $F_2(x_{h+1}) > 0$. We then choose the learning rate $\tau_{i,h+1}, \tau_{-i,h+1}$ such that

$$-f_{i,1}(x_{i,h+2}) < -f_{i,2}(x_{i,h+1}) - \beta_1 \tau_{i,h+1} \|F_2(x_{h+1})\|^2, \tag{50a}$$

$$-f_{-i,1}(x_{-i,h+2}) < -f_{-i,2}(x_{-i,h+1}) - \beta_1 \tau_{-i,h+1} \|F_2(x_{h+1})\|^2, \tag{50b}$$

Still, from (48), we know (50a) can't be true and (50b) is easy to realized by setting $\tau_{-i,h+1}$. Thus, the $x_{i,h+1}$ will not be updated and $x_{-i,h+1}$ will be updated and push the CP time back to 1. Following a similar analysis, we know $x_{-i,h+2} > x_{-i,h+1}$ and $x_{-i,h+2} < r_{-i}$ and are supposed to increase to reach the critical point in subsystem 2 until converge, which indicates the left part of the critical point in the objective function $-f_{-i,2}$, where its gradient $-f'_{-i,2} < 0$. Thus, we have $-f_{-i,2}(x_{-i,h+2}) < -f_{-i,2}(x_{-i,h+1})$.

Now, let's look at the entire trajectory, we have $x_{i,h+2} = x_{i,h+1} < x_{i,h}$ for agent i and $x_{-i,h+2} > x_{-i,h+1} = x_{-i,h}$ for agent $-i$, indicating

$$- f_{i,1}(x_{i,h+2}) = -f_{i,1}(x_{i,h+1}) < -f_{i,1}(x_{i,h}), \tag{51a}$$

$$- f_{-i,2}(x_{-i,h+2}) < -f_{-i,2}(x_{-i,h+1}) = -f_{-i,2}(x_{-i,h}). \tag{51b}$$

Considering trajectory starting from subsystem 2, switching to subsystem 1, then back to subsystem 2 can show the $-f_{i,2}(x_{i,h+2}) > -f_{i,2}(x_{i,h})$ and $-f_{-i,1}(x_{-i,h+2}) > -f_{-i,1}(x_{-i,h})$ following the similar analysis, we omit the redundant math.

Now, let's analyze if the trajectory starts from subsystem 1 and stays more steps in subsystem 2 before going back to subsystem 1. Given the switching pair $\underline{h}, \overline{h}$ as described in Theorem 7, i.e, $\underline{h} = \overline{h} = 1, \underline{h} < h < \overline{h}, h = 2$. For agent $-i$, staying in subsystem 2 gradually increases x_{-i} until it goes back to subsystem 1 or reaches the critical point of $-f_{-i,2}$ in subsystem 2, which means converging to the critical point. Similar to (51b), we have

$$x_{-i,\overline{h}} > x_{-i,\overline{h}-1} > ... > x_{-i,\underline{h}+1} = x_{-i,\underline{h}}, \tag{52a}$$

$$- f_{-i,2}(x_{-i,\overline{h}}) < -f_{-i,2}(x_{-i,\overline{h}-1}) < ... < -f_{-i,2}(x_{-i,\underline{h}+1}) = -f_{-i,2}(x_{-i,\underline{h}}). \tag{52b}$$

where the first inequality and last equality equality comes from (51b).

For agent i, although $x_{i,h}$ will increase in subsystem 2, the $x_{i,h}, x_{-i,h}$ still within subsystem 2, and we have

$$\max\{x_{i,h} + x_{-i,h} | h \in (\underline{h}, \overline{h})\} < \min\{x_{i,\overline{h}} + x_{-i,\overline{h}}, x_{i,\underline{h}} + x_{-i,\underline{h}}\} \tag{53a}$$

Because (50a) can't be true and (50b) is easily satisfied and $x_{-i,h}$ gradually increase in subsystem 2 as described in (52), the system switch must be activated by agent $-i$. This means

$$x_{-i,\underline{h}} < \max\{x_{-i,h}|h \in (\underline{h}, \overline{h})\} < x_{-i,\overline{h}}, \tag{53b}$$

Combined with (51a), we have

$$x_{i,\underline{h}} > \max\{x_{i,h}|h \in (\underline{h}, \overline{h})\} = x_{i,\overline{h}}, \tag{53c}$$

where the first inequality is obtained due to two cases: i) if $x_{i,\overline{h}} + x_{-i,\overline{h}} < x_{i,\underline{h}} + x_{-i,\underline{h}}$, and we know $x_{-i,\underline{h}} < x_{-i,\overline{h}}$ from (53b), thus, $x_{i,\underline{h}} > x_{i,\overline{h}}$; ii) if $x_{i,\overline{h}} + x_{-i,\overline{h}} \geq x_{i,\underline{h}} + x_{-i,\underline{h}}$, we know (53a) is equivalent to

$$\max\{x_{i,h} + x_{-i,h}|h \in (\underline{h}, \overline{h})\} < x_{i,\underline{h}} + x_{-i,\underline{h}}, \tag{53d}$$

and due to (53b), $x_{-i,\underline{h}} < \max\{x_{-i,h}|h \in (\underline{h}, \overline{h})\}$, thus, $x_{i,\underline{h}} > \max\{x_{i,h}|h \in (\underline{h}, \overline{h})\}$. Thus, according to (53c), we have

$$-f_{i,1}(x_{i,\overline{h}}) < -f_{i,1}(x_{i,\underline{h}}) \tag{53e}$$

This proves the Theorem by showing $-f_{i,1}, -f_{-i,2}$ reduce and $-f_{i,2}, -f_{-i,1}$ increase if $X_{i,1} + x_i > X_{i,2} - x_i$, following Theorem 6, we know the system will converge to either the switching surface $f_{i,1} + f_{-i,1} = f_{i,2} + f_{-i,2}$ or the critical points $-r_i$ ($F_1(x) = 0$) when $S_{1,0} \geq S_{2,0}$ or r_i ($F_2(x) = 0$) when $S_{1,0} < S_{2,0}$.

□

APPENDIX E
PROOF OF THEOREM 9

Proof. We first show the optimal solution from the centralized model as described in (15). Under the following condition, we can easily get both agents' solution as the critical point r_i, r_{-i} by first-order optimality condition.

$$-r_i - r_{-i} \leq b, \text{if } S_{1,0} \geq S_{2,0}, \tag{54a}$$

$$r_i + r_{-i} \geq b, \text{if } S_{1,0} < S_{2,0}. \tag{54b}$$

Otherwise, the system demand will be balanced in the two periods, i.e., $x_i + x_{-i} = b$, and we add the constraints with Lagrange multipliers λ

$$\mathcal{L}(x, \lambda) = \pi \max\{S_1(x), S_2(x)\} + \alpha_i x_i^2 + \alpha_{-i} x_{-i}^2 + \lambda(b - x_i - x_{-i}), \tag{54c}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \pm\pi + 2\alpha_i x_i - \lambda = 0, \tag{54d}$$

$$\frac{\partial \mathcal{L}}{\partial x_{-i}} = \pm\pi + 2\alpha_{-i} x_{-i} - \lambda = 0, \tag{54e}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_i + x_{-i} = b, \tag{54f}$$

$$x_i = \frac{\alpha_{-i}}{\alpha_i + \alpha_{-i}} b, x_{-i} = \frac{\alpha_i}{\alpha_i + \alpha_{-i}} b, \tag{54g}$$

where the sign of $\pm$ is determined by the system baseline demand $S_{1,0}, S_{2,0}$. Overall, the solution for the centralized model (15) is

$$x_i = \frac{\alpha_{-i} b}{\alpha_i + \alpha_{-i}}, x_{-i} = \frac{\alpha_i b}{\alpha_i + \alpha_{-i}}, \qquad \text{if } -r_i - r_{-i} \leq b \leq r_i + r_{-i} \tag{55a}$$

$$x_i = -r_i, x_{-i} = -r_{-i}, \qquad \text{if } -r_i - r_{-i} > b \tag{55b}$$

$$x_i = r_i, x_{-i} = r_{-i}, \qquad \text{if } r_i + r_{-i} < b \tag{55c}$$

Note that the peak shaving effectiveness defined in the theorem statement is equal to directly comparing the $x^*_{\text{cp},i} + x^*_{\text{cp},-i}$ with $x^*_{\text{cen},i} + x^*_{\text{cen},-i}$ because the baseline demand is the same in both models. Combine with the game solution in Theorem 5, under the concave game conditions, we have $x^*_{\text{cp},i} = x^*_{\text{cen},i} = \pm r_i, x^*_{\text{cp},-i} = x^*_{\text{cen},-i} = \pm r_{-i}$, indicating the peak shaving effectiveness at equilibrium equal to 1. Under the non-concave game and quasiconcave game conditions, it is easy to see both game model and centralized model will balance system demand, i.e., $x^*_{\text{cp},i} + x^*_{\text{cp},-i} = x^*_{\text{cen},i} + x^*_{\text{cen},-i} = b$, meaning that the peak shaving effectiveness at equilibrium also equal to 1. This proves the theorem.

□

APPENDIX F
PROOF OF THEOREM 10

Proof. Recall the centralized model solution from Theorem 9, the $x^*_{\text{cen},i}$ is the same under quasiconcave and non-concave game conditions, i.e.,

$$x^*_{\text{cen},i} = \frac{\alpha_{-i} b}{\alpha_i + \alpha_{-i}}, x^*_{\text{cen},-i} = \frac{\alpha_i b}{\alpha_i + \alpha_{-i}}. \tag{56}$$

Although the game solution is different under these two conditions, we can denote it as x^*_i, and according to the definition of PoA (19a), the PoA is

$$P = \frac{f_i(x^*) + f_{-i}(x^*)}{f_i(x^*_{\text{cen}}) + f_{-i}(x^*_{\text{cen}})} = \frac{\pi S + \alpha_i x_i^{*2} + \alpha_{-i} x_{-i}^{*2}}{\pi S + \alpha_i (\frac{\alpha_{-i} b}{\alpha_i + \alpha_{-i}})^2 + \alpha_{-i} (\frac{\alpha_i b}{\alpha_i + \alpha_{-i}})^2} \tag{57a}$$

By replacing $b = x^*_i + x^*_{-i}$, we have

$$P = \frac{\pi S + \alpha_i x_i^{*2} + \alpha_{-i} x_{-i}^{*2}}{\pi S + \frac{\alpha_i \alpha_{-i}^2 (x^*_i + x^*_{-i})^2 + \alpha_i^2 \alpha_{-i} (x^*_i + x^*_{-i})^2}{(\alpha_i + \alpha_{-i})^2}} = \frac{(\pi S + \alpha_i x_i^{*2} + \alpha_{-i} x_{-i}^{*2})(\alpha_i + \alpha_{-i})}{\pi S (\alpha_i + \alpha_{-i}) + \alpha_i \alpha_{-i} (x^*_i + x^*_{-i})^2}$$

$$= \frac{\pi S(\alpha_i + \alpha_{-i}) + \alpha_i\alpha_{-i}(x_i^{*2} + x_{-i}^{*2}) + (\alpha_i x_i^*)^2 + (\alpha_{-i}x_{-i}^*)^2}{\pi S(\alpha_i + \alpha_{-i}) + \alpha_i\alpha_{-i}(x_i^{*2} + x_{-i}^{*2}) + 2\alpha_i\alpha_{-i}x_i^* x_{-i}^*}, \tag{57b}$$

The first two terms in the denominator and nominator are the same, thus, the difference between denominator and nominator is

$$(x_i^*\alpha_i)^2 + (x_{-i}^*\alpha_{-i})^2 - 2x_i^* x_{-i}^*\alpha_i\alpha_{-i} = (x_i^*\alpha_i - x_{-i}^*\alpha_{-i})^2, \tag{58}$$

We then know P will increase with $(\alpha_i x_i^* - \alpha_{-i}x_{-i}^*)^2$, and P is continuous/differentiable function regarding α and x_i^*, thus $\partial P/\partial[(\alpha_i x_i^* - \alpha_{-i}x_{-i}^*)^2] > 0$. Note that the shifting cost is $\alpha_i x_i^{*2}$ and $\alpha_i x_i^* = \partial(\alpha_i x_i^{*2})/\partial x_i^*$, thus we call it marginal shifting cost. □

APPENDIX G
PROOF OF THEOREM 11

Proof. Under the concave game condition, from Theorems 5 and 9, if $S_{1,0} \geq S_{2,0}$, we have

$$x_i^* = -r_i, x_{-i}^* = -r_{-i}, x_{\text{cen},i}^* = -r_i, x_{\text{cen},-i}^* = -r_{-i}, \tag{59}$$

and change the critical point from $-r_i$ to r_i will get the solution when $S_{1,0} < S_{2,0}$. Thus, $x_i^* = x_{\text{cen},i}^*$ for $i, -i$ and $P = 1$, indicating concave game condition is equivalent to centralized condition.

From Theorem 10, we know the PoA under quasiconcave and non-concave game conditions can be written as (57b), and nominator minus denominator is $(\alpha_i x_i^* - \alpha_{-i}x_{-i}^*)^2 \geq 0$. Thus, $P \geq 1$ under these two conditions, indicating quasiconcave and non-concave games always cause higher anarchy than concave games.

Then Given fixed $\pi, S, \alpha > 0$, the PoA (57b) is only affected by the solution structure x_i^*, x_{-i}^*. Note that $X_{i,1}, X_{i,2}, X_{-i,1}, X_{-i,2}$ could be variant such that $X_{i,1} + X_{i,2} + X_{-i,1} + X_{-i,2} = 2S$, which may cause quasiconcave or non-concave game condition, i.e., all agent $i, -i$ satisfy the following or not,

$$-\frac{\pi}{2\alpha_i} \leq b_i = \frac{X_{i,2} - X_{i,1}}{2} \leq \frac{\pi}{2\alpha_i}. \tag{60}$$

We basically fixed the other parameters that appeared in the PoA expression of (57b) to explicitly show the influence of the game type change. According to Theorem 5, we know both quasiconcave and non-concave games balance the system demand in the two time periods, i.e., $x_i^* + x_{-i}^* = b$. For the quasiconcave game, $x_i^* = b_i, x_{-i}^* = b_{-i}$, and satisfy the (60), while for non-concave game, $x_i^* = \pm r_i, x_{-i}^* = b \mp r_i$, and $b_i > r_i$ or $b_i < -r_i$. Suppose $x_i^* = r_i, x_{-i}^* = b - r_i$, then the condition is $b_i > r_i$ and $b_{-i} < b - r_i$, indicating

$$(\alpha_i b_i - \alpha_{-i}b_{-i})^2 > (\alpha_i r_i - \alpha_{-i}(b - r_i))^2. \tag{61}$$

If $x_i^* = -r_i, x_{-i}^* = b + r_i$, the situation is similar and we omit the redundant math here. Thus, the quasiconcave game causes higher PoA than the non-concave game under fixed $\pi, S, \alpha > 0$. These finish the proof of the Theorem. □

APPENDIX H
PROOF OF PROPOSITION 12

Overview of the proof: We derive two lemmas to show the existence and uniqueness of NE in the multi-agent CP game G', respectively. We first show the CP game G' exists NE (Lemma 17) and show the NE exists in Lemma 17 is unique (Lemma 18). We first introduce two concepts.

(1) *Payoff security (Reny [38]).* Agent i can secure the payoff $f_i(x_i, x_{-i}) - \epsilon \in \mathbb{R}$ at $x \in \mathcal{X}$ iff for every $\epsilon > 0$, there exist a $\hat{x_i} \in \mathcal{X}_i$ such that $f_i(\hat{x_i}, x'_{-i}) \geq f_i(x_i, x_{-i}) - \epsilon$ for every x'_{-i} in some neighborhood of x_{-i}. Furthermore, we say that a game G' is payoff secure iff every player i can secure payoff for every $x_i \in \mathcal{X}_i$.

(2) *Diagonally strictly concave (Rosen [39]).* Define the pseudo-gradient of the sum of all players' payoff functions $\sum_{i \in N} f_i(x)$ with transport symbol ^{T} and total differential operator ∇ as

$$F(x) = [\nabla_1 f_1(x), \cdots, \nabla_N f_N(x)]^T. \tag{62a}$$

Then the function $\sum_{i \in N} f_i(x)$ is diagonally strictly concave for $x \in \mathcal{X}$ if for every $x_{\mathrm{a}}, x_{\mathrm{b}} \in \mathcal{X}$ we have

$$(x_{\mathrm{a}} - x_{\mathrm{b}})F(x_{\mathrm{b}}) + (x_{\mathrm{b}} - x_{\mathrm{a}})F(x_{\mathrm{a}}) > 0. \tag{62b}$$

Among them, payoff security means every agent can secure a payoff value in any strategy profile if they have a strategy that provides at least this value, even if other players slightly change their strategies. We also have the sufficient conditions for diagonally strictly concave function from Rosen [39], namely that the symmetric matrix $G(x) + G^T(x)$ be negative definite for $x \in \mathcal{X}$, where $G(x)$ is the Jacobian of $F(x)$ with respect to x. We then introduce the existence lemma.

Lemma 17. *Existence.* The multi-agent two-period CP game G' as described in (20) and (21b), has a pure-strategy NE as defined in Definition 1.

Proof. According to Reny [38], a compact, convex, bounded and quasiconcave game has a pure strategy NE if the sum of the player's payoff functions is u.s.c. as defined in Definition 1 in the whole strategy set and the game is payoff security as defined in Definition 1. Thus, we focus on (a) the sum of the player's payoff functions, i.e., $\sum_{i \in N} f_i(x)$, is u.s.c. in $x \in \mathcal{X}$ and (b) the game G' is payoff secure, to prove the existence of pure-strategy NE.

(a). By summing all player's payoff functions,

$$\sum_{i\in N} f_i(x) = -\pi \sum_{i\in N}(X_{i,1}+x_i)I(S_1-S_2) - \pi \sum_{i\in N}(X_{i,2}-x_i)I(S_2-S_1) - \sum_{i\in N}\alpha_i x_i^2$$
$$= \pi S_1 I(S_1-S_2) - \pi S_2 I(S_2-S_1) - \sum_{i\in N}\alpha_i x_i^2. \tag{63}$$

This function is u.s.c. according to Definition 1 due to $\pi S_1 = \pi S_2$ when $S_1 = S_2$.

(b). We then show that each player is payoff secure by analyzing their strategies. According to Definition 1, given any strategy pair (x_i, x_{-i}) and for any $\epsilon > 0$, agents $i, -i$ can secure the payoff $f_i(x_i, x_{-i}) - \epsilon$ and $f_{-i}(x_i, x_{-i}) - \epsilon$. Note that here $-i$ denotes all agents but i.

Suppose the CP time is $t = 1$, i.e., $S_1 \geq S_2$, if one of the agent $-i$ slightly increases x_{-i}, the CP time is still 1, and agent i's payoff is the same if agent i keeps the strategy profile x_i; if one of the agent $-i$ slightly decrease x_{-i}, CP time is possible to change, but agent i can secure the payoff $f_i(x_i, x_{-i}) - \epsilon$ by slightly reducing its strategy x_i. Basically, there exist a $\delta > 0$ small enough that satisfy $f_i(x_i + \delta, x'_{-i}) > f_i(x_i, x_{-i}) - \epsilon$ for all $S'_1 = X_{i,1} + x_i + \delta + \sum_{-i\in N}(X_{-i,1} + x'_{-i}) > S_2$. Thus, agents $i, i \in N$ are payoff secure, and CP time $t = 2$ follows the same analysis. Thus, the CP game is payoff secure.

According to the Reny [38], the CP game has a pure-strategy NE. □

Lemma 18. *Uniqueness.* The pure-strategy NE (x_i^*, x_{-i}^*) as described in Lemma 17 is unique and obtained when (23) hold.

Proof. According to Rosen's method [39], if the sum of the player's payoff function is diagonally strictly concave as defined in Definition 1, then the equilibrium point described in Lemma 17 is unique if the constraints in the strategy set are concave, i.e., $\mathcal{X} = \{x | h(x) \geq 0\}$, where $h(x)$ is a concave function.

We first study the uniqueness of pure-strategy NE in each subsystem 1 and 2 corresponding to CP time in 1 and 2, with the strategy set $\mathcal{X}_1, \mathcal{X}_2$. In strategy set $\mathcal{X}_1$, we have the weighted non-negative sum as defined in Definition 1, here we slightly abuse our notation by extending two agents $i, -i$ to N agents $i \in N$,

$$\sum_{i\in N} f_i(x) = -\pi \sum_{i\in N}(X_{i,1}+x_i) - \sum_{i\in N}\alpha_i x_i^2. \tag{64}$$

The Hessian H with respect to x is $G = G^T = -2\text{diag}(\alpha_1, \cdots, \alpha_N)$, where $\text{diag}(\cdot) : \mathbb{R}^N \to \mathbb{R}^{N\cdot N}$. This shows that $G + G^T$ is obviously negative definite, thus, we know the sum of the player's payoff function is diagonally strictly concave. This means, according to Rosen's method, there is a unique NE in the strategy set $\mathcal{X}_1$. Following the same analysis, we can conclude there is a unique NE in the sub-strategy set $\mathcal{X}_2$.

Now, the difficult part of proving this Lemma lies in the switching between two sub-strategy sets. Thus, we separate three cases according to the CP time to analyze the NE.

- Case (i): CP time is always 1, i.e., $S_1(x) \geq S_2(x), \forall x \in \mathcal{X}$;
- Case (ii): CP time is always 2, i.e., $S_1(x) < S_2(x), \forall x \in \mathcal{X}$.
- Case (iii): CP time changes during the gaming process, i.e., $\exists x', x'' \in \mathcal{X}, S_1(x') - S_2(x') \geq 0, S_1(x'') - S_2(x'') < 0$;

We first show *Cases (i) and (ii)* can't exist under the quasiconcave conditions (21b) because only one subsystem is active during the entire solution process and the game is concave. Taking case (i) for example, mathematically, we have, $S_1(x) \geq S_2(x), \forall x \in \mathcal{X} = \mathcal{X}_1$, i.e.,

$$S_1(x) = \sum_{i \in N}(X_{i,1} + x_i) \geq \sum_{i \in N}(X_{i,2} - x_i) = S_2(x), \tag{65a}$$

$$\frac{\sum_{i \in N}(X_{i,2} - X_{i,1})}{2} = b \leq \sum_{i \in N} x_i, \tag{65b}$$

which aligns with the conditions of the concave game. The same analysis can be applied to case (ii), where CP time is always 2. Thus, we know both cases (i) and (ii) can't exist under the quasiconcave conditions.

We then analyze *Case (iii)* and start with the two-agent setting with agent $i, -i$. We first note that the CP charge, either $\pi(X_{i,1} + x_i)$ or $\pi(X_{i,2} - x_i)$, is relatively greater compared with penalty $\alpha_i x_i^2$ for all agents because the CP time can change in the case (iii). From the best response perspective, suppose the CP time is 1, i.e., $S_1 = S_2 + \delta$, where δ is an infinitesimal number. Now, if agent i benefits by reducing x_i, which changes CP time to 2. This indicates the demand of agent i in time 1 is greater than time 2, i.e., $X_{i,1} + x_i \geq X_{i,2} - x_i$. Now if agent $-i$'s demand in time 1 is less than time 2, i.e., $X_{-i,1} + x_{-i} < X_{-i,2} - x_{-i}$, then CP time change from 1 to 2 harms agent $-i$'s benefit, so agent $-i$ want to push the CP time back to 1 by increasing x_{-i}; if $X_{-i,1} + x_{-i} \geq X_{-i,2} - x_{-i}$, CP time changes to 2 means agent $-i$ can benefit by shifting x_{-i} away from time 2, i.e., still increase x_{-i}. This means the best response of agent $-i$'s is always adversarial to that of agent i.

This shows agents are *fully competitive*; basically, if agent i can benefit from changing strategy, that strategy harms another agent's benefit. This is similar to the zero-sum game setting. Thus, we can apply the following min-max formulation to analyze the best responses,

$$x_i^* = \arg\max_{x_i}\min_{x_{-i}} f_i(x_i, x_{-i}) = \arg\max_{x_i}\begin{cases} -\pi(X_{i,1}+x_i) - \alpha_i x_i^2, \\ \qquad x_i \geq b_i, x_i \geq b - x_{-i}^* \\ -\pi(X_{i,2}-x_i) - \alpha_i x_i^2, \\ \qquad x_i < b_i, x_i < b - x_{-i}^* \end{cases}. \tag{66a}$$

where x_{-i}^* is the optimal solution for agent $-i$ following the same structure, and the first and second case corresponds to CP time 1 and 2. By applying the first-order optimality condition, the optimal solution of x_i is

$$x_i^* = \begin{cases} \max\{-r_i, b - x_{-i}^*, b_i\}, & S_1 \geq S_2 \\ \min\{r_i, b - x_{-i}^*, b_i\}, & S_1 < S_2 \end{cases}. \tag{66b}$$

The optimal solution for agent $-i$ follows the same structure

$$x_{-i}^* = \arg\max_{x_{-i}}\min_{x_i} f_{-i}(x_i, x_{-i}) = \begin{cases} \max\{-r_{-i}, b - x_i^*, b_{-i}\}, & S_1 \geq S_2 \\ \min\{r_{-i}, b - x_i^*, b_{-i}\}, & S_1 < S_2 \end{cases}. \tag{66c}$$

According to (5), $-r_i \leq b_i \leq r_i$, $-r_{-i} \leq b_{-i} \leq r_{-i}$, we know when x_i^*, x_{-i}^* reach optimal solution, if $S_1 \geq S_2$, and $x_{-i}^* = \max\{b - x_i^*, b_{-i}\}$. The same solution holds for x_i^*, i.e., $x_i^* = \max\{b - x_{-i}^*, b_i\}$. Then, according to (66b) and (66c), the only solutions are $x_i^* = b_i,\ x_{-i}^* = b_{-i}$, and the conditions are

$$X_{i,2} + X_{-i,2} - x_i^* - x_{-i}^* = S_2 = X_{i,1} + X_{-i,1} + x_i^* + x_{-i}^* = S_1. \tag{67}$$

Then, we use the two-agent solution to analyze the solution in the N-agent setting. Given that all agents are capable, define a *partition* of set N into two disjoint subsets, denoted as N_a, N_b, such that $N_\text{a} \cup N_\text{b} = N, N_\text{a} \cap N_\text{b} = \emptyset$, and the demand for these two subsets is unequal in at least one time period, i.e.,

$$\{\sum_{i\in N_\text{a}}(X_{i,1}+x_i) \neq \sum_{i\in N_\text{b}}(X_{i,1}+x_i)\} \cup \{\sum_{i\in N_\text{a}}(X_{i,2}-x_i) \neq \sum_{i\in N_\text{b}}(X_{i,2}-x_i)\}. \tag{68}$$

Then treat these two sets as two agents with strategy x_a, x_b and apply the two-agent solutions, we know the two sets will balance their demand in the two time periods, i.e., the following is true for sets N_a, N_b,

$$x_\text{a}^* = \frac{\sum_{i\in N_\text{a}} X_{i,2} - \sum_{i\in N_\text{a}} X_{i,1}}{2} = \frac{\sum_{i\in N_\text{a}}(X_{i,2} - X_{i,1})}{2} = \sum_{i\in N_\text{a}} b_i = \sum_{i\in N_\text{a}} x_i^*. \tag{69}$$

Then, applying the same partition to N_a, N_b to get the subsets $N_\text{a,a}, N_\text{a,b}$ and $N_\text{b,a}, N_\text{b,b}$, and applying the two-agent solutions, we still know the two subsets will balance demand in the two time periods, i.e., $\sum_{i\in N_\text{a,a}} x_i^* = \sum_{i\in N_\text{a,a}} b_i$ for sets $N_\text{a,a}, N_\text{a,b}, N_\text{b,a}, N_\text{b,b}$. Iterating applying the partition to each subset to get corresponding two subsets, the subset will finally become a singleton and we have $x_i^* = b_i, i \in N$, and

$$\sum_{i \in N}(X_{i,2} - x_i^*) = S_2 = \sum_{i \in N}(X_{i,1} + x_i^*) = S_1. \tag{70}$$

This shows the NE will always be obtained when $S_1 = S_2$ when the game is quasiconcave, i.e., all agents' are capable as described in Proposition 4. This also means during the game, subsystems 1 and 2 are active interactively, and the agent's strategy set switches between $\mathcal{X}_1$ and $\mathcal{X}_2$ and finally converge at the connecting point of both strategy set $S_1 = S_2$. This means there is a unique equilibrium point in the entire strategy set $\mathcal{X}$. Indeed, all agents will minimize the payment associated with S_1, S_2 imbalance as any imbalance results in a significant CP charge change caused by the opponent's strategy.

Thus, we conclude that the NE as described in (23) is unique and proves this Lemma. □

Proof of Proposition 12. We first show the quasiconcave CP game G' as described in (20) and (21b) has pure-strategy NE based on Lemma 17. Then, we prove the NE existed in Lemma 17 is unique based on Lemma 18 and the unique NE is obtained as described in (23). □

APPENDIX I
PROOF OF PROPOSITION 13

Proof. Given $S_{1,0} < S_{2,0}$. We first assume two virtual agents as CP agent and non-CP agent. We denote the strategy as x_{cp}, baseline demand as $X_{\text{cp},2} = \sum_{i \in N_{\text{cp}}} X_{i,2}, X_{\text{cp},1} = \sum_{i \in N_{\text{cp}}} X_{i,1}$, balance point $b_{\text{cp}} = (X_{\text{cp},2} - X_{\text{cp},1})/2$, and critical point $r_{\text{cp}} = \sum_{i \in N_{\text{cp}}} r_i$ for CP agent, and change the subscript to ncp for non-CP time agent. Note that for CP agent $X_{\text{cp},1} < X_{\text{cp},2}$ (the same definition with CP-time agent) and non-CP agent vice verse, we also have $b_{\text{cp}} + b_{\text{ncp}} = b$.

Then, from Theorem 5, we know these two virtual agents will balance the system demand at equilibrium. Depending on which agent reaches the critical point first, the best strategy for one agent is the critical point and the other is to balance the left unbalanced system demand. Suppose the CP agent reaches the critical point first, i.e., the demand shifting of the non-CP agent cannot be offset by the CP agent, meaning that the CP agent can't flatten the system demand given the non-CP agent's shifting even if the CP agent shifts the maximum amount of demand. We can express the condition as

$$r_{\text{cp}} < b_{\text{cp}}, -r_{\text{ncp}} - r_{\text{cp}} \le b \le r_{\text{ncp}} + r_{\text{cp}}, \tag{71a}$$

and the NE between these two virtual agents is

$$x_{\text{cp}}^* = \min\{r_{\text{cp}}, b_{\text{cp}}\} = r_{\text{cp}}, x_{\text{ncp}}^* = b - r_{\text{cp}}, \tag{71b}$$

We then aggregate CP-time agents and non-CP-time into the CP-time agent set and non-CP-time agent set and fixed the agent in each set, as the CP-time agent and non-CP-time agent are defined based on their baseline demand conditions. Obviously, the baseline demand, balance point, and critical point are equivalent between the set of CP agents and the virtual CP agent. However, they are not totally equivalent because their strategies are different. The strategy for the CP-time agent set is determined by each agent $x_i^*, i \in N_{\text{cp}}$. The non-CP-time agent set follows the same idea. Note that the same baseline demand and critical point indicate their best response rationale is the same; basically, the virtual CP agent benefits by shifting demand away from the CP time, and we have the same for the agents in the CP-time agent set.

From Theorem 5, we know each agent's maximum shifting capacity is b_i, i.e., $x_i^* = \min\{r_i, b_i\}$, and from the condition (71a), we know although the CP-time agent set and virtual CP agent have the same conditions, CP-time agent set can't reach the best strategy of the virtual CP agent due to the following

$$\sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\} < \sum_{i \in N_{\text{cp}}} r_i = r_{\text{cp}} = x_{\text{cp}}^* < \sum_{i \in N_{\text{cp}}} b_i. \tag{72}$$

Note that the best response of both the virtual CP agent and CP-time agent set is to shift demand away from the CP time, and the benefits they obtained monotonically increase with the shifting amount. From (72), we know the maximum shifting amount of the CP-time agent set is less than the best strategy of the virtual CP agent, indicating the best strategy for the CP-time agent set is their maximum shifting amount $\sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\}$.

We then verify in this case the non-CP-time agent set will still balance system demand to get profits. We first show the non-concave game condition still holds, which is equivalent between virtual agents and sets of agents, i.e.,

$$-\sum_{i \in N_{\text{ncp}}} r_i - \sum_{i \in N_{\text{cp}}} r_i = -\sum_{i \in N} r_i \leq b \leq \sum_{i \in N_{\text{ncp}}} r_i + \sum_{i \in N_{\text{cp}}} r_i = \sum_{i \in N} r_i \tag{73}$$

Thus, under the condition (72) with $\sum_{i \in N_{\text{cp}}} x_i^* = \sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\}$, the non-CP-time agent set can still balance system demand. Then, we show the best response rationale is the same between the non-CP-time agent set and the virtual non-CP agent due to the same baseline demand and critical point conditions. From Theorem 5, we know the virtual non-CP agent benefited by shifting demand to the CP time in response to the virtual CP agent's strategy. Thus, the benefit of the non-CP-time agent set increases with the shifting amount to the CP time because it offsets the shifting of the CP-time-agent set and further fills the system demand difference when the CP-time-agent set reaches maximum shifting capability. Also, due to $\sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\} < \sum_{i \in N_{\text{cp}}} r_i$, we have $b - \sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\} > b - \sum_{i \in N_{\text{cp}}} r_i$, meaning that the non-CP-time agent set can actually get more benefit than the virtual non-CP agent.

To conclude, the CP-time agents in the CP-time agent set with strategy $x_i^* = \min\{r_i, b_i\}, i \in N_{\text{cp}}$ form a best response to the non-CP-time agents in the non-CP-time agent set with strategy $\sum_{i \in N_{\text{ncp}}} x_i^* = b - \sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\}$ under the condition of

$$\sum_{i \in N_{\text{cp}}} \min\{r_i, b_i\} < \sum_{i \in N_{\text{cp}}} b_i, -\sum_{i \in N} r_i \leq b \leq \sum_{i \in N} r_i; \tag{74}$$

otherwise, the best strategy will be

$$x_i^* = \max\{-r_i, b_i\}, i \in N_{\text{ncp}}, \sum_{i \in N_{\text{cp}}} x_i^* = b - \sum_{i \in N_{\text{ncp}}} \max\{-r_i, b_i\}, \tag{75}$$

corresponding to the condition

$$\sum_{i \in N_{\text{ncp}}} \max\{-r_i, b_i\} > \sum_{i \in N_{\text{cp}}} b_i, -\sum_{i \in N} r_i \leq b \leq \sum_{i \in N} r_i. \tag{76}$$

Note that here, the set of agents is fixed, and we focus on the set level performance for the CP-time agent set and non-CP-time agent set. Specifically, when the aggregated (set-level) demand of the CP-time agent set is higher in the non-CP time, the CP-time agent set will perform as the non-CP-time agent set, and meanwhile, the aggregated demand of the non-CP-time agent set must be higher in the CP time, and the non-CP-time agent set perform as the CP-time agent set. This is similar to the two-agent case, where they only exchange strategy when the relative relationship between their own peak time and CP time changes.

□

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